SEQUENCE ENTROPY AND IT-TUPLES FOR MINIMAL GROUP ACTIONS

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Abstract. Let G be an infinite discrete countable group and (X, G) a minimal G-system. First, we prove that

$$h^*_{top}(X, G) \geq \log \sum_{\mu \in \mathcal{M}^e(X,G)} e^{h^*_\mu(X,G)},$$

where $h^*_{top}(X, G)$ and $h^*_\mu(X, G)$ are the supremum of the topological and metric sequence entropy, respectively. Additionally, if G is abelian, there exists $K \in \mathbb{N} \cup \{\infty\}$ with $\log K \leq h^*_{top}(X, G)$ such that it is a regular K-to-one extension of its maximal equicontinuous factor.

Furthermore, for any infinite countable discrete group G, we show that if the factor map from a minimal G-system to its maximal equicontinuous factor is regular K_1-to-one and almost K_2-to-one, then the system admits $\lceil K_1/K_2 \rceil$-IT-tuples, where $K_1 \in \mathbb{N} \cup \{\infty\}$ and $K_2 \in \mathbb{N}$. As a corollary, we refine the upper bound on the number of ergodic measures for systems that are almost N-to-one extensions of their maximal equicontinuous factors and lack K-IT-tuples, thereby improving the result of Huang et al. (2021, J. Funct. Anal. 280, no. 12, No. 109000).

1. Introduction

Throughout this paper, we always assume that G is an infinite discrete countable group, unless otherwise stated. Meanwhile, we always assume that X is a compact metric space with a metric d and $\mathcal{B}_X$ is the Borel σ-algebra of (X, d). By a *G-system*, we mean a pair (X, G), where G acts on X as a group of homeomorphisms. By a *G-measure preserving system* (G-mps for short) we mean a triple (X, μ, G), where (X, G) is a G-system and μ is a G-invariant Borel probability measure on X.

The notion of *sequence entropy* for a $\mathbb{Z}$-mps was introduced by Kushnirenko [28], who showed that an ergodic $\mathbb{Z}$-mps has discrete spectrum if and only if its sequence entropy is zero for any sequence. Goodman [13] later introduced *topological sequence entropy* for a $\mathbb{Z}$-system, and Huang, Li, Shao and Ye [17] proved a minimal $\mathbb{Z}$-system with zero topological sequence entropy along any sequence, is uniquely ergodic, has discrete spectrum and is an almost one-to-one extension to its maximal equicontinuous factor. For general group actions, the sequence entropy can be defined similarly. Huang and Ye [21] introduced the maximal pattern entropy, denoted by $h^*_{top}(X, G)$,

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and proved it equals to the supremum of topological sequence entropy over all sequences of G. From [17, 21], the maximal pattern entropy of a G-system is $\log K$ for some $K \in \mathbb{N} \cup \{\infty\}$. Correspondingly, in the measure-theoretical setting, we denote by $h^*_\mu(X,G)$ the supremum of metric sequence entropy of G-mps (X,μ,G) over all sequences of G. In this paper, we will show that for any ergodic G-mps (X,μ,G), there is $K \in \mathbb{N} \cup \{\infty\}$ such that $h^*_\mu(X,G) = \log K$ **(see Theorem 2.9)**, and clarify that this result does not hold for non-invariant measures **(see Proposition 2.10)**.

Denote by $\mathcal{M}^e(X,G)$ the set of ergodic measures of (X,G). It is well known that $\sup_{\mu\in\mathcal{M}^e(X,G)} h^*_\mu(X,G) \le h^*_{top}(X,G)$, where the inequality can hold strictly (see e.g. [21, Theorem 2.6]). If further assume the minimality, we have a better lower bound.

Theorem 1.1. *Let (X,G) be a minimal G-system. Define*

$$a := \begin{cases} \log \sum_{\mu\in\mathcal{M}^e(X,G)} e^{h^*_\mu(X,G)}, & \text{if } \mathcal{M}^e(X,G) \text{ is finite or countable,} \\ 0, & \text{if } \mathcal{M}^e(X,G) = \emptyset, \\ \infty, & \text{if } \mathcal{M}^e(X,G) \text{ is uncountable.} \end{cases}$$

Then, the following inequality holds:

$$h^*_{top}(X,G) \ge a. \tag{1.1}$$

In particular, if $\sup_{\mu\in\mathcal{M}^e(X,G)} h^*_\mu(X,G) = h^*_{top}(X,G) < \infty$, then it is uniquely ergodic.

Remark 1.2. *We provide an example showing that the equality in (1.1) may hold, demonstrating that this lower bound is sharp. In [10], the authors constructed a minimal system (X,G), which is an at most two-to-one and almost one-to-one extension of its maximal equicontinuous factor, and has two distinct ergodic invariant measures. By Theorem 4.4 in [21], we have $h^*_{top}(X,G) \le \log 2$. Applying Theorem 1.1, since the system has two distinct ergodic measures, we also have $h^*_{top}(X,G) \ge \log 2$. Therefore, $h^*_{top}(X,G) = \log 2$.*

As a direct corollary of Theorem 1.1, the number of ergodic measures of a minimal system is bounded by the maximal pattern entropy.

Corollary 1.3. *Let (X,G) be a minimal G-system. Then $|\mathcal{M}^e(X,G)| \le e^{h^*_{top}(X,G)}$.*

Let (X,G) be a G-system and $\pi : (X,G) \to (H,G)$ be the factor map to its maximal equicontinuous factor. Here π can be obtained in a constructive way via the regionally proximal equivalence relation [34]. Let ν be the Haar measure of H. We say (X,G) is a *regular K-to-one extension* of its maximal equicontinuous factor if $\nu(\{y \in H : |\pi^{-1}(y)| = K\}) = 1$.

Investigating the structure of minimal systems that lack non-trivial sequence entropy tuples is also an interesting problem, as explored in [30]. When G is abelian, Huang, Lian, Shao, and Ye [18] demonstrated if a minimal G-system (X,G) satisfies $h^*_{top}(X,G) = \log K$ for some $K \in \mathbb{N}$, then it is an almost K'-to-one extension of its maximal equicontinuous factor for some $K' \le K$. In a recent study, Qiu and Yu

[32] proved that if $G = \mathbb{Z}$, it is regular K-to-one extension of its maximal equicontinuous factor for some natural number K with $\log K \leq h^*_{top}(X, \mathbb{Z})$. However, their approach does not generalize to general infinite discrete countable abelian groups. In this paper, we employ a new method to extend this result to any infinite discrete countable abelian group.

Theorem 1.4. *Let (X, G) be a minimal G-system, where G is abelian. Then there exists $K \in \mathbb{N} \cup \{\infty\}$ with $\log K \leq h^*_{top}(X, G)$ such that (X, G) is regular K-to-one extension of its maximal equicontinuous factor.*

Remark 1.5. *For general minimal group actions Theorem 1.4 may not hold. For example, put $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$. Define T, S on $\mathbb{T}$ by $Tx = x + \alpha$ and $Sx = x^2$ for $x \in [0, 1)$ where α is an irrational number. Let G be the group generated by T and S. Since $(\mathbb{T}, T)$ is minimal, so is $(\mathbb{T}, G)$. The maximal equicontinuous factor of $(\mathbb{T}, G)$ is trivial and the size of the fiber is ∞. However, Glasner and Megrelishvili [12] showed that $(\mathbb{T}, G)$ is null.*

The local entropy theory has unfolded around the concept of entropy pair introduced by Blanchard [4]. From the viewpoint of independence, for a K-tuple $(A_1, \ldots, A_K)$ of subsets of X, we say that a set $J \subset G$ is an *independent set* for this tuple if for any nonempty finite subset $I \subset J$ and function $\sigma : I \to \{1, 2, \ldots, K\}$, $\cap_{s \in S} s^{-1} A_{\sigma(s)} \neq \emptyset$. A pair of two different points $(x_1, x_2) \in X \times X$ is called strongly scrambled by Huang [16] if, for any open neighborhood U_i of x_i, $i = 1, 2$, the pair (U_1, U_2) has an infinite independent set, which was used to study the tameness of systems. Kerr and Li [23] independently introduced the corresponding definition for tuples, and call them *K-IT-tuples*. If each component of a K-IT-tuple is pairwise distinct, we call it an *essential K-IT-tuple*.

Fuhrmann et al. [10] first showed that the inconsistency between the measure-theoretical and topological perspectives of the extension to its maximal equicontinuous factor guarantees the existence of IT-pairs. Huang et al. [18] extended this result to IT-tuples. More precisely, they proved that if π is almost one-to-one and regular K-to-one for some $K \in \mathbb{N}$ with $K \geq 2$, then (X, G) has essential K-IT-tuples. In the following theorem, we relax the almost one-to-one condition to an almost finite-to-one condition.

Theorem 1.6. *Let (X, G) be a minimal G-system and $\pi : (X, G) \to (H, G)$ be the factor map on the maximal equicontinuous factor of (X, G). Let ν be the Haar measure of H. Given $K_1 \in \mathbb{N} \cup \{\infty\}, K_2 \in \mathbb{N}$, suppose that (X, G) satisfies the following two conditions:*

(1) *π is regular K_1-to-one;*
(2) *π is almost K_2-to-one.*

Then (X, G) has essential K-IT-tuples for all $K \in \mathbb{N}$ with $2 \leq K \leq \lceil \frac{K_1}{K_2} \rceil$.

Remark 1.7. *From Theorem 1.6, if a minimal G-system (X, G) has no invariant measures and has no K-IT-tuples for some $K \in \mathbb{N}$ with $K \geq 2$, then π is almost ∞-to-one.*

Indeed, if this conclusion is not valid, there exists $K_2 \in \mathbb{N}$ such that π is almost K_2-to-one. Since (X, G) has no K-IT-tuples, then there exists $K_1 \in \mathbb{N}$ such that $\nu(\{y \in H : |\pi^{-1}(y)| = K_1\}) = 1$, where ν is the Haar measure on H. Then $\int_H \frac{1}{K_1} \sum_{x \in \pi^{-1}(y)} \delta_x d\nu(y)$ is an invariant measure which is a contradiction.

The example in Remark 1.5 clearly has no invariant measures. This, combined with the discussion in Remark 1.7, raises a natural question.

Question 1.8. *Assume a minimal G-system (X, G) has no essential K-IT-tuples for some $K \geq 2$ and $\mathcal{M}^e(X, G) \neq \emptyset$. Is it true that (X, G) is regular K' to one extension of its maximal equicontinuous factor for some natural number $K' \leq K - 1$?*

Remark 1.9. *When $K = 2$, this question is solved by Fuhrmann, Glasner, Jäger and Oertel [10], where they proved that if a tame (introduced by Köhler [25]) minimal system has at least one invariant measure then it is regular. Recall that a G-system is tame if and only if it has no essential 2-IT-tuples [23].*

Finally, let us end this section with a corollary of Theorem 1.6. In [18], Huang, Lian, Shao and Ye demonstrated that if a minimal system under an amenable group action is an almost N-to-one extension of its maximal equicontinuous factor and lacks K-IT-tuples, then it has no more than $N(K-1)^N$ different ergodic measures. We extend their result to infinite countable discrete groups, and refine the upper bound of the number of ergodic measures.

Theorem 1.10. *Let (X, G) be a minimal G-system and $\pi : (X, G) \to (H, G)$ be the factor map on the maximal equicontinuous factor of (X, G). Assume that (X, G) satisfies the following two conditions:*

1. *π is almost N-to-one for some $N \in \mathbb{N}$;*
2. *(X, G) has no K-IT-tuples for some $K \in \mathbb{N}, K \geq 2$.*

Then (X, G) has no more than $N(K-1)$ different ergodic measures.

The structure of the paper is as follows. In Section 2, we recall some basic notions and results. In Section 3, we provide the proof of Theorem 1.1. In Section 4, we prove Theorem 1.4, Theorem 1.6 and Theorem 1.10.

2. Preliminary

In this section, we review some basic notions and fundamental properties of G-systems.

2.1. **Regionally proximal relation.** A G-system (X, G) is called *minimal* if X contains no proper non-empty closed G-invariant subsets. It is easy to verify that a G-system is minimal if and only if every orbit is dense if and only if $\{gU : g \in G\}$ is an open cover of X for any nonempty open subset U of X. A *factor map* $\pi : X \to Y$ between two G-systems (X, G) and (Y, G) is a continuous onto map which intertwines the actions. In this case, we say that (Y, G) is a *factor* of (X, G) or (X, G)

is an *extension* of (Y,G). We say π is *almost N to one* for some $N\in\mathbb{N}\cup\{\infty\}$ if $\{y\in Y:|\pi^{-1}(y)|=N\}$ is a dense G_δ subset of Y. In the case (Y,G) is minimal, π is almost N to one if and only if $N=\min_{y\in Y}|\pi^{-1}(y)|$.

For $k\geq 2$, the *k-regionally proximal relation* [1] is defined by

$$\begin{aligned}RP_k(X,G)=\{(x_i)_{i=1}^k\in X^k:\ &\text{for any } \epsilon>0 \text{ there exist } x_i'\in X \text{ and } g\in G\\ &\text{with } d(x_i,x_i')\leq\epsilon(1\leq i\leq k),\ \text{and } d(gx_i',gx_j')\leq\epsilon(1\leq i\leq j\leq k)\}.\end{aligned}$$

A G-system (X,G) is *equicontinuous* if for any $\epsilon>0$, there is $\delta>0$ such that whenever $x,y\in X$ with $d(x,y)<\delta$, then $d(gx,gy)<\epsilon$ for all $g\in G$. There is a smallest invariant equivalence relation S_{eq} such that the quotient system $(X/S_{eq},G)$ is equicontinuous [9]. The equivalence relation S_{eq} is called the equicontinuous structure relation and the factor $(X/S_{eq},G)$ is called the *maximal equicontinuous factor* of (X,G). We remark that S_{eq} is the smallest closed G-invariant equivalent relation containing the 2-regionally proximal relation $RP_2(X,G)$. The 2-regional proximal relation $RP_2(X,G)$ is G-invariat, closed, symmetric and reflective. Of course, $RP_2(X,G)$ is an equivalence relation if and only if $S_{eq}=RP_2(X,G)$. It turns out that in many cases $RP_2(X,G)$ is an equivalence relation. For example, (X,G) is minimal and $X\times X$ has a dense set of minimal points [1, Theorem 8].

Theorem 2.1. *Let (X,G) be a minimal G-system and $k\geq 2$. Assume that $X\times X$ has a dense set of minimal points. If $x_1,x_2,\cdots,x_k\in X$ satisfy $(x_1,x_i)\in RP_2(X,G)$ for $1\leq i\leq k$, then $(x_1,x_2,\cdots,x_k)\in RP_k(X,G)$.*

Especially, if G is abelian and (X,G) is a minimal G-system, then $(x,gx)\in X\times X$ is minimal for all $x\in X$ and $g\in G$. Thus, $X\times X$ has a dense set of minimal points and then Theorem 2.1 is true in this case (In the case G is abelian, one can see [19] or [31] for other proofs).

2.2. **Independence set.** We recall the notion of independence sets introduced in [23, Definition 2.1](see also [24, Definition 8.7]).

Definition 2.2. *Let (X,G) be a G-system and $k\in\mathbb{N}$. For a tuple $A=(A_1,\cdots,A_k)$ of subsets of X, we say that a set $J\subset G$ is an independence set for A if for every nonempty finite subset $I\subset J$ and function $\sigma:I\to\{1,2,\cdots,k\}$,*

$$\cap_{s\in I}s^{-1}A_{\sigma(s)}\neq\emptyset.$$

Definition 2.3. *Let (X,G) be a G-system and $n\geq 2$. We call a tuple $(x_1,\cdots,x_n)\in X^n$ is an n-IT-tuple (or an IT-pair for $n=2$) if for any product neighborhood $U_1\times U_2\times\cdots\times U_n$ of $(x_1,\cdots,x_n)$ in X^n the tuple $(U_1,U_2,\cdots,U_n)$ has an infinite independence set. If each component of a n-IT-tuple is pairwise distinct, we call it an essential n-IT-tuple.*

We denote the set of n-IT-tuples and essential n-IT-tuples by $\mathrm{IT}_n(X,G)$ and $\mathrm{IT}_n^e(X,G)$, respectively.

The diagonal of X^n is defined by

$$\triangle_n(X) = \{(x, \cdots, x) \in X^n : x \in X\},$$

and put

$$\triangle^{(n)}(X) = \{(x_1, x_2, \cdots, x_n) \in X^n : \text{ for some } i \neq j, x_i = x_j\}.$$

When $n = 2$ one writes $\triangle(X) = \triangle_2(X) = \triangle^{(2)}(X)$.

The following results can be found in [23, Proposition 6.4].

Proposition 2.4. *Let (X, G) be a G-system and $n \geq 2$.*

1. *Let $(A_1, \cdots, A_n)$ be a tuple of non-empty closed subsets of X which has an infinite independence set. Then there exists an n-IT-tuple $(x_1, \cdots, x_n)$ with $x_j \in A_j$ for all $1 \leq j \leq n$.*
2. *$IT_2^e(X, G)$ is nonempty if and only if (X, G) is non-tame.*
3. *$IT_n(X, G)$ is a closed G-invariant subset of X^n.*
4. *Let $\pi : (X, G) \to (Y, G)$ be a factor map. Then $\pi^{(n)}(IT_n(X, G)) = IT_n(Y, G)$, where $\pi^{(n)} : X^n \to Y^n$ defined by $\pi^{(n)}(x_1, \cdots, x_n) = (\pi(x_1), \cdots, \pi(x_n))$.*
5. *Suppose that Z is a closed G-invariant subset of X. Then $IT_n(Z, G) \subset IT_n(X, G)$.*

2.3. **Ellis semigroup.** An *Ellis semigroup* is a compact space with a semigroup multiplication which is continuous in only one variable. For an Ellis semigroup E, the element u with $u^2 = u$ is called an *idempotent*. The Ellis-Numakura theorem says that for any Ellis semigroup E, the set $J(E)$ of idempotents of E is not empty [8]. An idempotent $u \in J(E)$ is *minimal* if $v \in J(E)$ and $vu = v$ implies $uv = u$.

Given a G-system (X, G), the Ellis semigroup $E(X)$ associated to (X, G) is defined as the closure of $\{x \mapsto tx : t \in G\} \subset X^X$ in the product topology, where the semi-group operation is given by the composition. On $E(X)$, we may consider the G-system given by $E(X) \ni s \mapsto ts$ for each element $t \in G$.

Theorem 2.5 ([2], pp. 52-53)**.** *Suppose H is a compact metric space and (H, G) is minimal and equicontinuous. Then $E(H)$ is a compact metrisable topological group. Further, we have the following.*

(a) *If G is abelian, then $E(H)$ is abelian and (H, G) is isomorphic to $(E(H), G)$.*
(b) *For a general G, (H, G) is a factor of $(E(H), G)$, where the factor map π is given by*

$$\pi : E(H) \to H, t \to th$$

for some fixed $h \in H$. Moreover, π is open.

In this paper, we use the notation $\mathrm{cl}(\cdot)$ to denote the topological closure of a set and $\mathrm{int}(\cdot)$ to denote its topological interior. The following is in [18, Proposition 3.2] which was proved to be true for $k = 2$ in [10, Proposition 3.3].

Proposition 2.6. *Let H be a locally compact second countable Hausdorff topological group with left Haar measure ν, and let $k \in \mathbb{N}$ with $k \geq 2$. Suppose that $V_1, \cdots, V_k \subset H$ are compact subsets that satisfy*

(1) $\mathrm{cl}(\mathrm{int}(V_i)) = V_i$ *for* $i = 1, 2, \cdots, k$*;*
(2) $\mathrm{int}(V_i) \cap \mathrm{int}(V_j) = \emptyset$ *for all* $1 \le i < j \le k$*;*
(3) $\nu(\cap_{i=1}^{k} V_i) > 0$.

Further, assume that $G \subset H$ *is a dense subgroup and* $\mathcal{G} \subset H$ *is a residual set. Then there exists an infinite set* $I \subset G$ *such that for all* $a \in \{1, 2, ..., k\}^I$ *there exists* $h \in \mathcal{G}$ *with the property that*

$$h \in \bigcap_{t \in I} t^{-1} int(V_{a_t}).$$

Now we introduce another type of Ellis semigroups.

Definition 2.7. *A filter on* G *is a nonempty family* $\mathfrak{p}$ *of subsets of* G *with the following properties:*

(1) *If* $A, B \in \mathfrak{p}$ *then* $A \cap B \in \mathfrak{p}$*;*
(2) *If* $A \in \mathfrak{p}$ *and* $A \subset B \subset G$*, then* $B \in \mathfrak{p}$*;*
(3) $\emptyset \notin \mathfrak{p}$.

If in addition, $\mathfrak{p}$ *satisfies that for any* $A \subset G$*, either* $A \in \mathfrak{p}$ *or* $G \setminus A \in \mathfrak{p}$*, then it is called an ultrafilter on* G [14, Section 3].

Let

$$\beta G := \{\mathfrak{p} : \mathfrak{p} \text{ is an ultrafilter on } G\}.$$

The sets with the form $\hat{H} := \{\mathfrak{p} \in \beta G : H \in \mathfrak{p}\}$ with $H \subset G$ form a base for a topology on βG. With this topology βG becomes a compact Hausdorff space. The semigroup operation on G is defined by

$$A \in \mathfrak{p} \cdot \mathfrak{q} \Leftrightarrow \{x \in G : Ax^{-1} \in \mathfrak{p}\} \in \mathfrak{q},$$

where $Ax^{-1} := \{y \in G : yx \in A\}$. Then βG is an Ellis semigroup.

Let $(U_g)_{g \in G}$ be a unitary representation of a group G on a Hilbert space $\mathcal{H}$, and

$$\mathcal{H}_c = \{\varphi \in \mathcal{H} : \{U_g \varphi : g \in G\} \text{ is precompact in } \mathcal{H}\}.$$

Since the unit ball in $\mathcal{H}$ is a compact metrizable space with respect to the weak topology, the expression $\mathfrak{p}\text{-}\lim_{g \in G} U_g \varphi$ has a well defined meaning. The following consequence was proved by Bergelson [3, Theorem 4.4 and Theorem 4.5].

Theorem 2.8. *Let* $(U_g)_g \in G$ *be a unitary representation of a group* G *on a Hilbert space* $\mathcal{H}$*. Let* $\mathfrak{p} \in \beta G$ *be a minimal idempotent. Then*

$$\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{wm},$$

where $\mathcal{H}_{wm} := \{\varphi \in \mathcal{H} : \mathfrak{p}\text{-}\lim_{g \in G} U_g \varphi = 0\}$.

2.4. **Measure decomposition.** Let X be a compact metric space. Denote $\mathcal{B}_X$ by the Borel σ-algebra of X and denote $\mathcal{M}(X)$ by the set of all Borel probability measures on X. For $\mu \in \mathcal{M}(X)$, denote by $\mathcal{B}_X^\mu$ the completion of $\mathcal{B}_X$ under μ and denote by $\mathrm{supp}(\mu)$ the support of μ, i.e. the smallest closed subset of X with full measure. Under the weak* topology, $\mathcal{M}(X)$ is a nonempty compact convex space. For a sub-σ-algebra $\mathcal{F}$ of $\mathcal{B}_X^\mu$, μ can be disintegrated over $\mathcal{F}$ as $\mu = \int_X \mu_{\mathcal{F},x} d\mu(x)$, where $\mu_{\mathcal{F},x} \in \mathcal{M}(X)$. The disintegration is characterized by the properties (2.1) and (2.2) below:

$$\text{for every } f \in L^1(X, \mathcal{B}_X^\mu, \mu), f \in L^1(X, \mathcal{B}_X^\mu, \mu_{\mathcal{F},x}) \text{ for } \mu\text{-a.e. } x \in X, \tag{2.1}$$

and the map $x \mapsto \int_X f d\mu_{\mathcal{F},x}$ is in $L^1(X, \mathcal{F}, \mu)$;

$$\text{for every } f \in L^1(X, \mathcal{B}_X^\mu, \mu), \mathbb{E}_\mu(f|\mathcal{F})(x) = \int_X f d\mu_{\mathcal{F},x} \text{ for } \mu\text{-a.e. } x \in X. \tag{2.2}$$

If $\mathcal{F}', \mathcal{F}$ are sub-σ-algebras of $\mathcal{B}_X^\mu$ with $\mathcal{F}' \subset \mathcal{F}$ then for every $f \in L^1(X, \mathcal{B}_X^\mu, \mu)$,

$$\mathbb{E}_\mu(f|\mathcal{F}')(x) = \mathbb{E}_\mu(\mathbb{E}_\mu(f|\mathcal{F})|\mathcal{F}')(x) \text{ for } \mu\text{-a.e. } x \in X.$$

This is equivalent to

$$\int_X f d\mu_{\mathcal{F}',x} = \int_X \int_X f d\mu_{\mathcal{F},y} d\mu_{\mathcal{F}',x}(y) \text{ for all } f \in L^1(X, \mathcal{B}_X^\mu, \mu) \text{ for } \mu\text{-a.e. } x \in X.$$

Then

$$\mu_{\mathcal{F}',x} = \int_X \mu_{\mathcal{F},y} d\mu_{\mathcal{F}',x}(y) \text{ for } \mu\text{-a.e. } x \in X. \tag{2.3}$$

Let (X, G) be a G-system. We say $\mu \in \mathcal{M}(X)$ is *G-invariant* if $\mu(g^{-1}B) = \mu(B)$ holds for all $B \in \mathcal{B}_X^\mu$ and $g \in G$. Denote by $\mathcal{M}(X, G)$ the set of G-invariant Borel probability measures of (X, G). We say $B \in \mathcal{B}_X^\mu$ is G-invariant if $g^{-1}B = B$ for all $g \in G$, and $\mu \in \mathcal{M}(X, G)$ is *ergodic* if for any G-invariant Borel set $B \in \mathcal{B}_X^\mu$, $\mu(B) = 0$ or $\mu(B) = 1$ holds. Denote by $\mathcal{M}^e(X, G)$ the set of ergodic measures of (X, G). Since G is countable, it is well known that $\mathcal{M}^e(X, G)$ is the collection of all extreme points of $\mathcal{M}(X, G)$. Using Choquet representation theorem, for each $\mu \in \mathcal{M}(X, G)$ there is a unique measure τ on the Borel subsets of the compact space $\mathcal{M}(X, G)$ such that $\tau(\mathcal{M}^e(X, G)) = 1$ and $\mu = \int_{\mathcal{M}^e(X,G)} m d\tau(m)$, which is called the *ergodic decomposition* of μ. If we let $\mathcal{I}_\mu(G) = \{B \in \mathcal{B}_X^\mu : \mu(gB \Delta B) = 0, \forall g \in G\}$, the ergodic decomposition of μ is just the disintegration of μ over $\mathcal{I}_\mu(G)$.

2.5. **Sequence entropy.** Let X be a compact metric space. A cover of X is a finite family of subsets of X whose union is X. A partition of X is a cover of X whose elements are pairwise disjoint. Denote by $\mathcal{C}_X$ (resp. $\mathcal{C}_X^o$) the set of all finite covers (resp. finite open covers) of X and by $\mathcal{P}_X$ the set of all finite partitions of X.

Let (X, G) be a G-system. Given a sequence $\mathcal{A} = (g_n)_{n=1}^\infty$ of G and $\mathcal{U} \in \mathcal{C}_X^o$, the *topological sequence entropy of $\mathcal{U}$ with respect to (X, G) along $\mathcal{A}$* is defined by

$$h_{top}^{\mathcal{A}}(G, \mathcal{U}) = \limsup_{n \to \infty} \frac{1}{n} \log N(\bigvee_{i=1}^n g_i^{-1}\mathcal{U}),$$

where $N(\bigvee_{i=1}^n g_i^{-1}\mathcal{U})$ is the minimal cardinality among all cardinalities of sub-covers of $\bigvee_{i=1}^n g_i^{-1}\mathcal{U}$. The *topological sequence entropy of* (X,G) *along* $\mathcal{A}$ is

$$h_{top}^{\mathcal{A}}(X,G) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} h_{top}^{\mathcal{A}}(G,\mathcal{U}).$$

Define the *maximal pattern entropy* of (X,G) by

$$h_{top}^*(X,G) = \sup_{\mathcal{A}} h_{top}^{\mathcal{A}}(X,G),$$

where the supremum is taken over all sequences of G.

By the ideas of Bowen [5] and Dinaburg [6], the topological sequence entropy along $\mathcal{A} = (g_n)_{n=1}^{\infty}$ can also be defined through the separated set. A subset E of X is a (n,ϵ)*-separated set along* $\mathcal{A}$ if for any distinct $x,y\in E$ there is $j\in\mathbb{N}$ with $1\le j\le n$ such that $d(g_jx,g_jy)>\epsilon$. Let $Sep_n^{\mathcal{A}}(\epsilon)$ be the largest cardinality of any (n,ϵ)-separated set along $\mathcal{A}$. Then

$$h_{top}^{\mathcal{A}}(X,G) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Sep_n^{\mathcal{A}}(\epsilon).$$

For $\mu\in\mathcal{M}(X)$, define

$$\mathcal{P}_X^{\mu} = \{\alpha\in\mathcal{P}_X : \text{each element in } \alpha \text{ belongs to } \mathcal{B}_X^{\mu}\}.$$

Given $\alpha\in\mathcal{P}_X^{\mu}$ and a sub-σ-algebra $\mathcal{A}$ of $\mathcal{B}_X^{\mu}$, define

$$H_\mu(\alpha|\mathcal{A}) = \sum_{A\in\alpha}\int_X -\mathbb{E}_\mu(1_A|\mathcal{A})\log\mathbb{E}_\mu(1_A|\mathcal{A})d\mu,$$

where $\mathbb{E}_\mu(1_A|\mathcal{A})$ is the conditional expectation of 1_A with respect to $\mathcal{A}$. One standard fact is that $H_\mu(\alpha|\mathcal{A})$ increases with respect to α and decreases with respect to $\mathcal{A}$. Set $\mathcal{N}=\{\emptyset,X\}$ and denote

$$H_\mu(\alpha) = H_\mu(\alpha|\mathcal{N}) = -\sum_{A\in\alpha}\mu(A)\log\mu(A).$$

Let (X,μ,G) be a G-mps, and $\alpha\subset\mathcal{P}_X^{\mu}$. The *sequence entropy of* α *with respect to* (X,μ,G) *along* $\mathcal{A}$ is

$$h_\mu^{\mathcal{A}}(G,\alpha) = \limsup_{n\to\infty}\frac{1}{n}H_\mu(\vee_{i=1}^n g_i^{-1}\alpha).$$

Define

$$h_\mu^*(G,\alpha) = \sup_{\mathcal{A}} h_\mu^{\mathcal{A}}(G,\alpha),$$

where the supremun is taken over all sequences of G. Define the *maximal pattern entropy of* (X,μ,G) by

$$h_\mu^*(G) = \sup_{\alpha\in\mathcal{P}_X^{\mu}} h_\mu^{\mathcal{A}}(G).$$

For $\mu\in\mathcal{M}(X,G)$, define the *Kronecker σ-algebra* of (X,μ,G) by

$$\mathcal{K}_\mu(G) = \{B\in\mathcal{B}_X^{\mu} : \{U_g1_B : g\in G\} \text{ is precompact in } L^2(\mu)\},$$

where $U_g f(x) = f(gx)$ for $g \in G, f \in L^2(\mu), x \in X$. It is easy to see that $\mathcal{K}_\mu(G)$ is a G-invariant sub-σ-algebra of $\mathcal{B}_X^\mu$. For $\mu \in \mathcal{M}^e(X,G)$, let $\mu = \int_X \mu_x d\mu(x)$ be the disintegration of μ over $\mathcal{K}_\mu(G)$. If μ is ergodic, there is $k_\mu \in \mathbb{N} \cup \{\infty\}$ such that $|\mathrm{supp}(\mu_x)| = k_\mu$ for μ-a.e. $x \in X$.

The following theorem shows the relation between sequence entropy and Kronecker σ-algebra. Its proof follows a similar approach to that of [20] for $G = \mathbb{Z}$, with the key difference being the replacement of natural number density by filters. Therefore, we provide the full proof in Appendix B. A similar result for amenable group actions can be found in [29].

Theorem 2.9. *Let (X,μ,G) be an ergodic G-mps. Let $\mu = \int_X \mu_x d\mu(x)$ be the disintegration of μ over $\mathcal{K}_\mu(G)$. Then $h_\mu^*(X,G) = \log k_\mu$.*

The following result shows the assumption of ergodicity in Theorem 2.9 is necessary.

Proposition 2.10. *There exists a minimal zero-dimensional $\mathbb{Z}$-system $(X_u,\mathbb{Z})$ such that, for any $r \in [0,\infty]$, there exists an invariant measure μ_r on $(X_u,\mathbb{Z})$ satisfying $h_{\mu_r}^*(X_u,\mathbb{Z}) = r$.*

Proof. By Weiss [37] (also Downarowicz [7]), there exists a minimal zero-dimensional $\mathbb{Z}$-system $(X_u,\mathbb{Z})$ carrying isomorphic copies of all nonatomic invariant measures. Now we prove for any $r \in [0,\infty]$, there exists $\mu \in \mathcal{M}(X_u,\mathbb{Z})$ satisfying $h_\mu^*(X_u,\mathbb{Z}) = r$.

Given $r \in [0,\infty]$, there exist $k_1, k_2 \in \mathbb{N} \cup \{\infty\}$ and $t \in [0,1]$ such that $r = t\log k_1 + (1-t)\log k_2$. Let $(X,\mu_1,\mathbb{Z})$ and $(Y,\mu_2,\mathbb{Z})$ be two ergodic mpss with $h_{\mu_1}^*(X,\mathbb{Z}) = \log k_1$ and $h_{\mu_2}^*(Y,\mathbb{Z}) = \log k_2$. We may further assume that μ_1 and μ_2 are nonatomic. Let $(Z,\mu,\mathbb{Z})$ be the mps such that Z is a disjoint union of X and Y, and $\mu = t\mu_1 + (1-t)\mu_2$. By a standard computation, we obtain that

$$h_\mu^*(Z,\mathbb{Z}) = t\log k_1 + (1-t)\log k_2 = r.$$

Since μ_1 and μ_2 are nonatomic, it follows that μ is also nonatomic. Thus, there exists μ_r such that $(X_u,\mu_r,\mathbb{Z})$ is isomorphic to $(Z,\mu,\mathbb{Z})$, and so $h_{\mu_r}^*(X_u,\mathbb{Z}) = r$. $\square$

3. Proof of Theorem 1.1

In the first part of this section, we present a new approach (Proposition 3.1) for estimating a lower bound on the maximum pattern entropy. This lays the groundwork for the proof of Theorem 1.1. In the second part, we proceed with the proof of Theorem 1.1.

Given the length and complexity of the proof, we provide an outline to enhance readability. First, we introduce the method for estimating a lower bound on topological sequence entropy.

Proposition 3.1. *Let (X,G) be a G-system, $\gamma \in (0,1)$ and $K \in \mathbb{N} \setminus \{1\}$. Assume that there is $\epsilon > 0$ such that for any $M \in \mathbb{N}$ and nonempty open subsets*

$U_1, U_2, \cdots, U_M$ of X there exist $g \in G$, $\mathcal{M} \subset \{1, 2, \cdots, M\}$ and $x_{m,k} \in U_m$ for $k \in \{1, 2, \cdots, K\}$, $m \in \mathcal{M}$ such that

(1) $|\mathcal{M}| \geq (1-\gamma)M$;
(2) $d(gx_{m,k}, gx_{m,k'}) > \epsilon$, for $1 \leq k < k' \leq K, m \in \mathcal{M}$.

Then $h^*_{top}(X, G) \geq \log((1-\gamma)K)$.

When $\mathcal{M}^e(X, G) = \emptyset$, Theorem 1.1 holds trivially. In the case where $\mathcal{M}^e(X, G) \neq \emptyset$, it suffices to show that for any $K < \log \sum_{\mu \in \mathcal{M}^e(X,G)} e^{h^*_\mu(X,G)}$, one has $h^*_{top}(X, G) > K$. For all $\epsilon > 0$, define

$$X^K_{>\epsilon} = \{(x_1, x_2, \cdots, x_K) \in X^K : d(x_k, x_{k'}) > \epsilon \text{ for } 1 \leq k < k' \leq K\}.$$

By Proposition 3.1, it remains to show that for any $\gamma > 0$, $M \in \mathbb{N}$ and nonempty open subsets $U_1, U_2, \cdots, U_M$ of X there exist $g \in G$, $\mathcal{M} \subset \{1, 2, \cdots, M\}$ with $|\mathcal{M}| \geq (1-\gamma)M$ and a tuple $(\bar{x}_1, \cdots, \bar{x}_M) \in U_1^K \times \cdots \times U_M^K$ such that

$$g\bar{x}_m \in X^K_{>\epsilon} \text{ for all } m \in \mathcal{M}.$$

By Birkhoff's pointwise ergodic theorem, to complete the proof, it suffices to construct an ergodic measure ν on $(X^K)^M$ satisfying

$$\nu(U_1^K \times \cdots \times U_M^K) > 0 \text{ and } \nu(P_1 \times \cdots \times P_M) > 0,$$

where $P_m = X^K$ if $m \notin \mathcal{M}$ and $P_m = X^K_{>\epsilon}$ if $m \in \mathcal{M}$.

3.1. **Preliminary results for Theorem 1.1.** First, we prove Proposition 3.1.

Proof of Proposition 3.1. For any nonempty finite subset E of G, denote by $\mathrm{Sep}_E(\epsilon)$ the maximal number L such that there exist $x_1, x_2, \cdots, x_L \in X$ with

$$\max_{g \in E} d(gx_l, gx_{l'}) > \epsilon, \text{ for all } 1 \leq l < l' \leq L.$$

We are going to show that there is a sequence $\mathcal{A} = \{g_1, g_2, \cdots\}$ of G such that $\mathrm{Sep}_{\{g_1, g_2, \cdots, g_n\}}(\epsilon) \geq (K(1-\gamma))^{n-1}$ for $n \in \mathbb{N}$.

Let $g_1 \in G$. It is clear that $\mathrm{Sep}_{\{g_1\}}(\epsilon) \geq 1 = (K(1-\gamma))^{1-1}$. Now assume that we have $g_1, g_2, \cdots, g_n \in G$ for some $n \in \mathbb{N}$ with $\mathrm{Sep}_{\{g_1, g_2, \cdots, g_n\}}(\epsilon) \geq (K(1-\gamma))^{n-1}$. Then there are nonempty open subsets $U_1, U_2, \cdots, U_{\mathrm{Sep}_{\{g_1, g_2, \cdots, g_n\}}(\epsilon)}$ of X such that

$$\max_{1 \leq i \leq n} d(g_iU_m, g_iU_{m'}) > \epsilon, \text{ for } 1 \leq m < m' \leq \mathrm{Sep}_{\{g_1, g_2, \cdots, g_n\}}(\epsilon),$$

where $d(U, V) := \inf\{d(x, y) : x \in U, y \in V\}$ for $U, V \subset X$. By assumption, there exist $g_{n+1} \in G$, $\mathcal{M} \subset \{1, 2, \cdots, \mathrm{Sep}_{\{g_1, g_2, \cdots, g_n\}}(\epsilon)\}$ and $x_{m,k} \in U_m$ for $k \in \{1, 2, \cdots, K\}$, $m \in \mathcal{M}$ such that

(1) $|\mathcal{M}| \geq (1-\gamma)\mathrm{Sep}_{\{g_1, g_2, \cdots, g_n\}}(\epsilon)$;
(2) $d(g_{n+1}x_{m,k}, g_{n+1}x_{m,k'}) > \epsilon$, for $1 \leq k < k' \leq K, m \in \mathcal{M}$.

Then for $k, k' \in \{1, 2, \cdots, K\}, m, m' \in \mathcal{M}$ with $(m, k) \neq (m', k')$, one has

$$\max_{1 \leq i \leq n+1} d(g_i x_{m,k}, g_i x_{m',k'}) > \epsilon.$$

This implies $\mathrm{Sep}_{\{g_1, g_2, \cdots, g_n, g_{n+1}\}}(\epsilon) \geq (K(1-\gamma))^n$. By induction, we get the required sequence $\mathcal{A} = \{g_1, g_2, \cdots\}$ of G. Therefore

$$h^*_{top}(X, G) \geq h^{\mathcal{A}}_{top}(X, G) \geq \limsup_{n \to \infty} \frac{\log \mathrm{Sep}_{\{g_1, g_2, \cdots, g_n\}}(\epsilon)}{n} \geq \log(K(1-\gamma)).$$

This ends the proof of Proposition 3.1. $\square$

To prove Theorem 1.1, we also need some lemmas.

Lemma 3.2. *Let $k \in \mathbb{N} \setminus \{1\}$ and $\epsilon > 0$. Let $L_i, i = 1, 2, \cdots, k$ be subsets of X such that for each i, $|L_i| = k$ and $d(x, y) > 2\epsilon$ for distinct $x, y \in L_i$. Then there exists $x_i \in L_i$, $i = 1, 2, \cdots, k$, such that $d(x_i, x_{i'}) > \epsilon$, for $1 \leq i < i' \leq k$.*

Proof. If $k = 2$, we let $x_1 \in L_1$. Since $|L_2| = 2, d(x, y) > 2\epsilon$, for all $x, y \in L_2$ with $x \neq y$, there is $x_2 \in L_2$ such that $d(x_1, x_2) > \epsilon$. Hence the lemma is true for $k = 2$.

Now we assume that the lemma is true for some $k \geq 2$. Then for $k + 1$, there must exist $L'_i \subset L_i, i \in \{1, 2, \cdots, k\}$ such that

$$|L'_i| = k, d(x, y) > 2\epsilon, \text{ for } x, y \in L'_i, x \neq y.$$

By induction, there is $x_i \in L'_i, i \in \{1, 2, \cdots, k\}$ such that

$$d(x_i, x_{i'}) > \epsilon, \text{ for } 1 \leq i < i' \leq k.$$

Note that $|L_{k+1}| = k + 1$. Then there exists $x_{k+1} \in L_{k+1}$, such that

$$d(x_i, x_{i'}) > \epsilon, \text{ for } 1 \leq i < i' \leq k + 1.$$

We finish the proof of Lemma 3.2. $\square$

For $\delta > 0$ and $K \in \mathbb{N} \setminus \{1\}$, put

$$X^K_{>\delta} = \{(x_1, x_2, \cdots, x_K) \in X^K : d(x_k, x_{k'}) > \delta \text{ for } 1 \leq k < k' \leq K\}.$$

Lemma 3.3. *Assume that $\mu \in \mathcal{M}(X)$ and $K \in \mathbb{N} \setminus \{1\}$. If $|supp(\mu)| \geq K$, then $\lim_{\delta \to 0} \mu^K(X^K_{>\delta}) > 0$.*

Proof. Since $|\mathrm{supp}(\mu)| \geq K$, we can find distinct points $x_1, x_2, \cdots, x_K \in \mathrm{supp}(\mu)$. Put

$$r := \min_{1 \leq k < k' \leq K} d(x_k, x_{k'}) > 0.$$

Then each $B(x_k, r/3)$ has positive measure and $B(x_1, \frac{r}{3})\times B(x_2, \frac{r}{3})\times\cdots\times B(x_K, \frac{r}{3}) \subset X^K_{>\delta}$ for all $\delta < \frac{r}{3}$. Therefore

$$\begin{aligned}
&\lim_{\delta\to 0}\mu^K(X^K_{>\delta})\\
&\geq \mu^K(B(x_1, r/3)\times B(x_2, r/3)\times\cdots\times B(x_K, r/3))\\
&= \Pi_{k=1}^K \mu(B(x_k, r/3))\\
&> 0.
\end{aligned}$$

This ends the proof of Lemma 3.3. □

Lemma 3.4. *Let $\mu \in \mathcal{M}(X)$. Let $M \in \mathbb{N}$, $\gamma \in (0,1)$ and $A_1, A_2, \cdots, A_M$ be measurable subsets of X with $\mu(A_i) \geq 1-\gamma$ for $i = 1, 2, \cdots, M$. Then there exists $\mathcal{M} \subset \{1, 2, \cdots, M\}$ such that*

1. *$|\mathcal{M}| \geq (1-\gamma)M$;*
2. *$\mu(\cap_{m\in\mathcal{M}} A_m) > 0$.*

Proof. Notice that

$$1-\gamma \leq \frac{1}{M}\sum_{m=1}^M \mu(A_m) = \int_X \frac{1}{M}\sum_{m=1}^M 1_{A_m}(x)d\mu(x).$$

It follows that the measurable subset $W = \{x \in X : \frac{1}{M}\sum_{m=1}^M 1_{A_m}(x) \geq 1-\gamma\}$ of X has positive measure. Notice that

$$W = \bigcup_{\substack{\mathcal{M}\subset\{1,2,\cdots,M\}\\ |\mathcal{M}|\geq(1-\gamma)M}} \bigcap_{m\in\mathcal{M}} A_m.$$

Since the set $\{\mathcal{M} \subset \{1, 2, \cdots, M\} : |\mathcal{M}| \geq (1-\gamma)M\}$ is finite, we can find a $\mathcal{M} \subset \{1, 2, \cdots, M\}$ such that $\mu(\cap_{m\in\mathcal{M}} A_m) > 0$ and $|\mathcal{M}| \geq (1-\gamma)M$. This ends the proof of Lemma 3.4. □

For $\mu \in \mathcal{M}(X, G)$, recall

$$\mathcal{I}_\mu(G) = \{B \in \mathcal{B}^\mu_X : \mu(B\Delta(gB)) = 0, \text{ for all } g \in G\},$$

and the Kronecker σ-algebra of (X, μ, G) is defined by

$$\mathcal{K}_\mu(G) = \{B \in \mathcal{B}^\mu_X : \{U_g 1_B : g \in G\} \text{ is precompact in } L^2(\mu)\}.$$

The following lemma 3.5 can be proved by an argument similar to that of [15, Theorem 16, pp. 53], and hence it will be proved in Appendix A.

Lemma 3.5. *Let (X, G) be a G-system. Then for $I \in \mathbb{N}$ and $\mu_1, \mu_2, \cdots, \mu_I \in \mathcal{M}(X, G)$, one has $\mathcal{I}_{\mu_1\times\mu_2\times\cdots\times\mu_I}(G) \subset \mathcal{K}_{\mu_1}(G)\times\mathcal{K}_{\mu_2}(G)\times\cdots\times\mathcal{K}_{\mu_I}(G)$.*

3.2. **Proof of Theorem 1.1.** Now we are going to prove Theorem 1.1. In fact, Theorem 1.1 immediately follows from Theorem 3.6 and Theorem 2.9.

Theorem 3.6. *Let (X,G) be a minimal G-system. Then*
$$h^*_{top}(X,G) \geq \log \sum_{\mu\in\mathcal{M}^e(X,G)} k_\mu,$$
where k_μ is defined as in Theorem 2.9.

Proof. If $\mathcal{M}^e(X,G)=\emptyset$, it is obviously true. Now we assume that $\mathcal{M}^e(X,G)\neq\emptyset$. Fix $I\in\mathbb{N}$ with $I\leq|\mathcal{M}^e(X,G)|$. Assume that $\mu_1,\cdots,\mu_I\in\mathcal{M}^e(X,G)$ are different ergodic measures of (X,G). Let $\mu_i=\int_X \mu_{i,x}\mu_i(x)$ be the disintegration of μ_i over $\mathcal{K}_{\mu_i}(G)$. To prove Theorem 3.6, it is enough to show that $h^*_{top}(X,G)\geq\log\sum_{i=1}^I k_i$ for natural numbers $k_i\leq k_{\mu_i}$, $i=1,2,\cdots,I$.

Now we fix natural numbers $k_i\leq k_{\mu_i}$ for $i=1,2,\cdots,I$. Put $\overline{X}=X^{\sum_{i=1}^I k_i}$ and $\overline{\mu}=\mu_1^{k_1}\times\mu_2^{k_2}\times\cdots\times\mu_I^{k_I}$. For $\overline{x}\in\overline{X}$, we denote
$$\overline{x}=(x_1,x_2,\cdots,x_{\sum_{i=1}^I k_i}) \text{ or } \overline{x}=(x_{1,1},\cdots,x_{1,k_1},\cdots,x_{I,k_I})$$
since there is no confusion. For $\overline{x}\in\overline{X}$, we put
$$\overline{\mu}_{\overline{x}}=\mu_{1,x_{1,1}}\times\cdots\times\mu_{1,x_{1,k_1}}\times\cdots\times\mu_{I,x_{I,k_I}}.$$
For $\epsilon>0$, put
$$W_\epsilon=\{\overline{x}\in\overline{X}:\overline{\mu}_{\overline{x}}(X^{\sum_{i=1}^I k_i}_{>\epsilon})>0\}. \tag{3.1}$$
For $j\in\mathbb{N}$, we let $i(j)$ be the number i such that
$$m\sum_{s=1}^I k_s+\sum_{s=1}^{i-1}k_s+1\leq j\leq m\sum_{s=1}^I k_s+\sum_{s=1}^i k_s \text{ for some } m\in\mathbb{N}\cup\{0\}.$$

Let $\mathcal{E}$ be the set of $\tau\in\mathcal{M}(\overline{X},G)$ with marginal $\mu_{i(j)}$ on the j-th coordinate for $1\leq j\leq\sum_{s=1}^I k_s$. For $\tau\in\mathcal{E}$ we put
$$\tau_{\mathcal{K}}=\int_{\overline{X}}\overline{\mu}_{\overline{x}}d\tau(\overline{x}).$$
Here $\tau_{\mathcal{K}}$ is well defined for $\tau\in\mathcal{E}$ since the map $x\mapsto\mu_{i,x}$ is measurable for each i. We have the following claim.

Claim 3.7. $\lim_{\epsilon\to0}\inf_{\tau\in\mathcal{E}}\tau_{\mathcal{K}}(W_\epsilon)=1$.

The proof of Claim 3.7 will be presented later. Now fix $0<\gamma<1$. By Claim 3.7, we can fix $\epsilon>0$ such that
$$\inf_{\tau\in\mathcal{E}}\tau_{\mathcal{K}}(W_\epsilon)\geq1-\gamma. \tag{3.2}$$
We are going to verify the conditions in Proposition 3.1. Fix $M\in\mathbb{N}$ and nonempty open subsets $U_1,U_2,\cdots,U_M$ of X. Let
$$\overline{\mu}^M=\int_{\overline{X}^M}\overline{\mu}_{\overline{x}_1,\overline{x}_2,\cdots,\overline{x}_M}d\overline{\mu}^M(\overline{x}_1,\overline{x}_2,\cdots,\overline{x}_M)$$

be the disintegration of $\overline{\mu}^M$ over $\mathcal{I}_{\overline{\mu}^M}(G)$ as well as the ergodic decomposition of $\overline{\mu}^M$. By Lemma 3.5,

$$\mathcal{I}_{\overline{\mu}^M}(G) \subset (\mathcal{K}_{\mu_1}(G)^{k_1} \times \mathcal{K}_{\mu_2}(G)^{k_2} \times \cdots \times \mathcal{K}_{\mu_I}(G)^{k_I})^M.$$

Then by (2.3), for $\overline{\mu}^M$-a.e. $(\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_M) \in \overline{X}^M$, one has

$$\overline{\mu}_{\overline{x}_1,\overline{x}_2,\cdots,\overline{x}_M} = \int_{\overline{X}^M} \overline{\mu}_{\overline{y}_1} \times \cdots \times \overline{\mu}_{\overline{y}_M} d\overline{\mu}_{\overline{x}_1,\overline{x}_2,\cdots,\overline{x}_M}(\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_M).$$

Since (X, G) is minimal, $\mathrm{supp}(\overline{\mu}^M) = X^{M(\sum_{i=1}^I k_i)}$. One can choose

$$\nu := \overline{\mu}_{\overline{z}_1,\overline{z}_2,\cdots,\overline{z}_M} \in \mathcal{M}^e(\overline{X}^M, G) \text{ for some } (\overline{z}_1, \overline{z}_2, \cdots, \overline{z}_M) \in \overline{X}^M$$

such that

$$\nu(U_1^{\sum_{i=1}^I k_i} \times U_2^{\sum_{i=1}^I k_i} \times \cdots \times U_M^{\sum_{i=1}^I k_i}) > 0,$$

$$\nu = \int_{\overline{X}^M} \overline{\mu}_{\overline{y}_1} \times \cdots \times \overline{\mu}_{\overline{y}_M} d\nu(\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_M), \tag{3.3}$$

and the projection of ν on the j-th coordinate is $\mu_{i(j)}$.

For $m \in \{1, 2, \cdots, M\}$, put

$$A_m = \overline{X}^{m-1} \times W_\epsilon \times \overline{X}^{M-m}.$$

Let p_m be the projection of $\overline{X}^M$ on the m-th $\overline{X}$. Then by (3.3),

$$p_m(\nu) = \int_{\overline{X}} \overline{\mu}_{\overline{y}} d\nu(\overline{y}).$$

By (3.2),

$$\nu(A_m) = p_m(\nu)(W_\epsilon) > 1 - \gamma, \text{ for all } m \in \{1, 2, \cdots, M\}.$$

By Lemma 3.4, there is $\mathcal{M} \subset \{1, 2, \cdots, M\}$ such that

(1) $|\mathcal{M}| \geq (1 - \gamma)M$;
(2) $\nu(\cap_{m\in\mathcal{M}} A_m) > 0$.

Put $P_m = \overline{X}$ if $m \notin \mathcal{M}$ and $P_m = X_{>\epsilon}^{\sum_{i=1}^I k_i}$ if $m \in \mathcal{M}$. By (3.1), for any $(\overline{y}_1, \cdots, \overline{y}_M) \in \cap_{m\in\mathcal{M}} A_m$,

$$\overline{\mu}_{\overline{y}_1} \times \cdots \times \overline{\mu}_{\overline{y}_M}(P_1 \times \cdots \times P_M) = \Pi_{m=1}^M \overline{\mu}_{\overline{y}_m}(P_m) > 0.$$

Since $\nu(\cap_{m\in\mathcal{M}} A_m) > 0$, one has

$$\begin{aligned}
&\nu(P_1 \times \cdots \times P_M) \\
&\geq \int_{\cap_{m\in\mathcal{M}} A_m} \overline{\mu}_{\overline{y}_1} \times \cdots \times \overline{\mu}_{\overline{y}_M}(P_1 \times \cdots \times P_M) d\nu(\overline{y}_1, \cdots, \overline{y}_M) \\
&> 0.
\end{aligned}$$

Since ν is ergodic and $\nu(U_1^{\sum_{i=1}^I k_i} \times U_2^{\sum_{i=1}^I k_i} \times \cdots \times U_M^{\sum_{i=1}^I k_i}) > 0$, there is $h \in G$ such that

$$(h(U_1^{\sum_{i=1}^I k_i} \times U_2^{\sum_{i=1}^I k_i} \times \cdots \times U_M^{\sum_{i=1}^I k_i})) \cap (P_1 \times P_2 \times \cdots \times P_M) \neq \emptyset.$$

We can find $(\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_M) \in U_1^{\sum_{i=1}^I k_i} \times U_2^{\sum_{i=1}^I k_i} \times \cdots \times U_M^{\sum_{i=1}^I k_i}$ such that

$$h(\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_M) \in P_1 \times P_2 \times \cdots \times P_M.$$

If we rewrite $\overline{x}_m = (x_{m,k})_{k=1,2,\cdots,\sum_{i=1}^I k_i}$ for $m \in \mathcal{M}$, we have

$$d(hx_{m,k}, hx_{m,k'}) > \epsilon, \text{ for } m \in \mathcal{M}, 1 \leq k < k' \leq \sum_{i=1}^I k_i$$

since $\overline{x}_m \in P_m = X_{>\epsilon}^{\sum_{i=1}^I k_i}$ for $m \in \mathcal{M}$. Since $x_{m,k} \in U_m$, for $m \in \mathcal{M}, 1 \leq k \leq \sum_{i=1}^I k_i$, the conditions in Proposition 3.1 are valid. Then by Proposition 3.1, $h^*_{top}(X, G) \geq \log((1-\gamma)\sum_{i=1}^I k_i)$. Let $\gamma \searrow 0$. One has $h^*_{top}(X, G) \geq \log \sum_{i=1}^I k_i$. This ends the proof of Theorem 3.6 by the beginning argument. □

Now we prove Claim 3.7.

Proof of Claim 3.7. Since $\mu_1, \mu_2, \cdots, \mu_I$ are different ergodic measures of (X, G), we can find $\epsilon_0 > 0$ and $r_\epsilon \searrow 0$ as $\epsilon \to 0$ and nonempty compact subsets $V_{i,\epsilon}$ of X for $i \in \{1, 2, \cdots, I\}, \epsilon \in (0, \epsilon_0)$ such that

$$d(V_{i,\epsilon}, V_{i',\epsilon}) > \epsilon \text{ for } 1 \leq i < i' \leq I \text{ and } \epsilon \in (0, \epsilon_0), \tag{3.4}$$

and

$$\mu_i(V_{i,\epsilon}) > 1 - r_\epsilon \text{ for } 1 \leq i \leq I \text{ and } \epsilon \in (0, \epsilon_0).$$

Then

$$\lim_{\epsilon \to 0} \mu_i(V_{i,\epsilon}) = 1 \text{ for } 1 \leq i \leq I. \tag{3.5}$$

For $i \in \{1, 2, \cdots, I\}$ with $k_i = 1$ and $\epsilon \in (0, \epsilon_0), \delta > 0$, put

$$V_{i,\epsilon,\delta} = V_{i,\epsilon} \cap \{x \in X : \mu_{i,x}(V_{i,\epsilon}) > 0\}, \tag{3.6}$$

where the subscript δ is simply for consistency with subsequent notations. Since $\mu_{i,x}(V_{i,\epsilon}) > 0$ for μ_i-a.e. $x \in V_{i,\epsilon}$, by (3.5), it follows that

$$\lim_{\delta \to 0} \liminf_{\epsilon \to 0} \mu_i(V_{i,\epsilon,\delta}) = \lim_{\epsilon \to 0} \mu_i(V_{i,\epsilon}) = 1. \tag{3.7}$$

For $i \in \{1, 2, \cdots, I\}$ with $k_i \geq 2$ and $\epsilon \in (0, \epsilon_0), 0 < \delta < 1$, put

$$W_{i,\epsilon,\delta} = \{x \in X : \mu_{i,x}^{k_i}(X_{>2\epsilon}^{k_i}) > 1 - (1-\delta)^{k_i}\}. \tag{3.8}$$

Since $|\text{supp}(\mu_{i,x})| \geq k_i > 1$ for μ_i-a.e. $x \in X$, one has by Lemma 3.3 that

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} \mu_i(W_{i,\epsilon,\delta}) = 1. \tag{3.9}$$

Put

$$V'_{i,\epsilon,\delta} = \{x \in X : \mu_{i,x}(V_{i,\epsilon}) > 1 - \delta\}. \tag{3.10}$$

and

$$V_{i,\epsilon,\delta} = V'_{i,\epsilon,\delta} \cap W_{i,\epsilon,\delta}. \tag{3.11}$$

Notice that

$$\begin{aligned}
\mu_i(V_{i,\epsilon}) &= \int_X \mu_{i,x}(V_{i,\epsilon})d\mu_i(x) \\
&= \int_{V'_{i,\epsilon,\delta}} \mu_{i,x}(V_{i,\epsilon})d\mu_i(x) + \int_{X\setminus V'_{i,\epsilon,\delta}} \mu_{i,x}(V_{i,\epsilon})d\mu_i(x) \\
&\le \mu_i(V'_{i,\epsilon,\delta}) + (1-\delta)\mu_i(X\setminus V'_{i,\epsilon,\delta}) \\
&= 1-\delta\mu_i(X\setminus V'_{i,\epsilon,\delta}).
\end{aligned}$$

This implies

$$\mu_i(X\setminus V'_{i,\epsilon,\delta}) \le \frac{1}{\delta}(1-\mu_i(V_{i,\epsilon})) \le \frac{1}{\delta}r_\epsilon.$$

Hence by (3.11),

$$\mu_i(V_{i,\epsilon,\delta}) \ge \mu_i(W_{i,\epsilon,\delta}) - \mu_i(X\setminus V'_{i,\epsilon,\delta}) \ge \mu_i(W_{i,\epsilon,\delta}) - \frac{1}{\delta}r_\epsilon.$$

By (3.9), one has

$$\lim_{\delta\to 0}\liminf_{\epsilon\to 0} \mu_i(V_{i,\epsilon,\delta}) = 1. \tag{3.12}$$

Now we are going to show that $V^{k_1}_{1,\epsilon,\delta}\times V^{k_2}_{2,\epsilon,\delta}\times\cdots\times V^{k_I}_{I,\epsilon,\delta}\subset W_\epsilon$. Without loss of generality, suppose that $V^{k_1}_{1,\epsilon,\delta}\times V^{k_2}_{2,\epsilon,\delta}\times\cdots\times V^{k_I}_{I,\epsilon,\delta}\neq\emptyset$. Fix a

$$\overline{x} = (x_{1,1},\cdots,x_{1,k_1},\cdots,x_{I,k_I}) \in V^{k_1}_{1,\epsilon,\delta}\times V^{k_2}_{2,\epsilon,\delta}\times\cdots\times V^{k_I}_{I,\epsilon,\delta}.$$

For $i\in\{1,2,\cdots,I\}$ with $k_i\ge 2$ and $k\in\{1,2,\cdots,k_i\}$, by (3.10) and (3.11),

$$\mu^{k_i}_{i,x_{i,k}}(V^{k_i}_{i,\epsilon}) > (1-\delta)^{k_i}.$$

Combining with (3.8), one can find k_i points in $V_{i,\epsilon}\cap\operatorname{supp}(\mu_{i,x_{i,k}})$ such that the distance between them are larger than 2ϵ. By Lemma 3.2, one can choose $y_{i,k}\in V_{i,\epsilon}\cap\operatorname{supp}(\mu_{i,x_{i,k}})$ for $1\le k\le k_i$ such that the distance between them are larger than ϵ. For $i\in\{1,2,\cdots,I\}$ with $k_i=1$, by (3.6) there is $y_{i,k_i}\in V_{i,\epsilon}\cap\operatorname{supp}(\mu_{i,x_{i,k_i}})$. Combining with (3.4), the distance between $y_{1,1},\cdots,y_{1,k_1},\cdots,y_{I,k_I}$ are larger than ϵ. This implies $(y_{1,1},\cdots,y_{1,k_1},\cdots,y_{I,k_I})\in\operatorname{supp}(\overline{\mu}_{\overline{x}})\cap X^{\sum_{i=1}^I k_i}_{>\epsilon}$. Since $X^{\sum_{i=1}^I k_i}_{>\epsilon}$ is a nonempty open subset of $X^{\sum_{i=1}^I k_i}$, one has $\overline{\mu}_{\overline{x}}(X^{\sum_{i=1}^I k_i}_{>\epsilon})>0$ and then $\overline{x}\in W_\epsilon$. Therefore $V^{k_1}_{1,\epsilon,\delta}\times V^{k_2}_{2,\epsilon,\delta}\times\cdots\times V^{k_I}_{I,\epsilon,\delta}\subset W_\epsilon$.

By (3.7) and (3.12), we can choose $\epsilon_n\searrow 0, \delta_n\searrow 0$ as $n\to\infty$ such that

$$\lim_{n\to\infty}\mu_i(V_{i,\epsilon_n,\delta_n}) = 1 \text{ for } i\in\{1,2,\cdots,I\}.$$

Then

$$\begin{aligned}
&\liminf_{\epsilon\to 0}\inf_{\tau\in\mathcal{E}}\tau_{\mathcal{K}}(W_\epsilon)\\
&\geq \liminf_{n\to\infty}\inf_{\tau\in\mathcal{E}}\tau_{\mathcal{K}}((V_{1,\epsilon_n,\delta_n})^{k_1}\times(V_{2,\epsilon_n,\delta_n})^{k_2}\times\cdots\times(V_{I,\epsilon_n,\delta_n})^{k_I})\\
&\geq \liminf_{n\to\infty}\inf_{\tau\in\mathcal{E}}\left(\tau_{\mathcal{K}}(\overline{X})-\sum_{j=1}^{\sum_{i=1}^I k_i}\tau_{\mathcal{K}}(X^{j-1}\times(X\setminus V_{i(j),\epsilon_n,\delta_n})\times X^{\sum_{i=1}^I k_i-j})\right)\\
&=\liminf_{n\to\infty}\left(1-\sum_{j=1}^{\sum_{i=1}^I k_i}\mu_{i(j)}(X\setminus V_{i(j),\epsilon_n,\delta_n})\right)\\
&=1.
\end{aligned}$$

This ends the proof of Claim 3.7. $\square$

4. Proofs of Theorem 1.4, Theorem 1.6 and Theorem 1.10

In this section, we are going to prove Theorem 1.4, Theorem 1.6 and Theorem 1.10.

4.1. Proof of Theorem 1.4.

Lemma 4.1. *Let (X,G) be a minimal G-system. Let $M\in\mathbb{N}$, $\delta>0$ and $(x_1,x_2,\cdots,x_M)\in RP_M(X,G)$. Let U be a nonempty open subset of X. Then there exist $h\in G$ and $x_m'\in B(x_m,\delta)$ such that $hx_m'\in U$ for $m=1,2,\cdots,M$.*

Proof. Since (X,G) is minimal, $\{gU:g\in G\}$ is an open cover of X. Let $\gamma<\delta$ be the Lebesgue number of $\{gU:g\in G\}$. Since $(x_1,x_2,\cdots,x_M)\in RP_M(X,G)$, there exists $h'\in G$ and $x_m'\in B(x_m,\gamma)$for $m=1,2,\cdots,M$ such that

$$d(h'x_m',h'x_{m'}')<\gamma, 1\leq m\leq m'\leq M.$$

There is $g\in G$ such that

$$h'x_m'\in gU \text{ for } 1\leq m\leq M.$$

This implies $g^{-1}h'x_m'\in U$ for $1\leq m\leq M$. Then $h=g^{-1}h'$ are as required. $\square$

Now we prove Theorem 1.4.

Proof of Theorem 1.4. Let G be an infinite discrete countable abelian group. Let (X,G) be a G-system and $\pi:(X,G)\to(H,G)$ be the factor map on the maximal equicontinuous factor of (X,G). Let ν be the Haar measure of H.

Since (H,ν,G) is ergodic, there is $P\in\mathbb{N}\cup\{\infty\}$ such that $|\pi^{-1}(y)|=P$ for ν-a.e. $y\in H$. Hence the set $\{y\in H:|\pi^{-1}(y)|\geq K\}$ has measure 0 or 1 for all $K\in\mathbb{N}$. Fix $K\in\mathbb{N}$ with $\nu(\{y\in H:|\pi^{-1}(y)|\geq K\})=1$. To prove Theorem 1.4, it is enough to show that $h_{top}^*(X,G)\geq\log K$.

Fix $\gamma \in (0,1)$. There is $\epsilon > 0$ such that the set

$$W := \{y \in H : \mathrm{Sep}(\pi^{-1}(y), 2\epsilon) \geq K\} \tag{4.1}$$

with $\nu(W) > 1-\gamma$, where $\mathrm{Sep}(\pi^{-1}(y), 2\epsilon)$ is the maximal number L such that there exist $x_1, x_2, \cdots, x_L \in \pi^{-1}(y)$ with $d(x_l, x_{l'}) > 2\epsilon$ for all $1 \leq l < l' \leq L$.

Now we are going to verify the conditions in Proposition 3.1. Fix $M \in \mathbb{N}$ and nonempty open subsets $U_1, U_2, \cdots, U_M$ of X. Since (X,G) is minimal, there is a nonempty open subset U of X and $g_1, g_2, \cdots, g_M \in G$ such that

$$g_m U \subset U_m, \text{ for all } m \in \{1, 2, \cdots, M\}.$$

Notice that

$$\int_H \sum_{m=1}^{M} 1_W(g_m y) d\nu(y) = M\nu(W) \geq (1-\gamma)M.$$

Thus, there is $y_0 \in H$ such that

$$|\{m \in \{1, 2, \cdots, M\} : g_m y_0 \in W\}| \geq (1-\gamma)M.$$

Put

$$\mathcal{M} = \{m \in \{1, 2, \cdots, M\} : g_m y_0 \in W\}.$$

By (4.1), there exists $z_{m,k} \in \pi^{-1}(g_m y_0), m \in \mathcal{M}, k = 1, 2, \cdots, K$ such that

$$d(z_{m,k}, z_{m,k'}) > 2\epsilon, \text{ for } m \in \mathcal{M}, 1 \leq k < k' \leq K.$$

Fix $\delta > 0$ such that for all $x, x' \in X$,

$$d(x, x') < \delta \text{ implies } d(g_m x, g_m x') < \frac{\epsilon}{2}, \text{ for all } m \in \mathcal{M}. \tag{4.2}$$

Notice that

$$\{g_m^{-1} z_{m,k} : m \in \mathcal{M}, k \in \{1, 2, \cdots, K\}\} \subset \pi^{-1}(y_0).$$

Since G is abelian and (X,G) is minimal, one has $\pi^{-1}(y_0) \times \pi^{-1}(y_0) \subset RP_2(X,G)$, which together with Theorem 2.1, implies $(g_m^{-1} z_{m,k})_{m \in \mathcal{M}, k \in \{1,2,\cdots,K\}} \in RP_{|\mathcal{M}|K}(X,G)$. By Lemma 4.1, there exists $h \in G$ and $z'_{m,k} \in B(g_m^{-1} z_{m,k}, \delta)$ such that

$$h z'_{m,k} \in U, \text{ for } m \in \mathcal{M}, 1 \leq k \leq K.$$

Put $x_{m,k} = g_m h z'_{m,k}$, for $m \in \mathcal{M}, 1 \leq k \leq K$. Then

$$x_{m,k} = g_m h z'_{m,k} \in g_m U \subset U_m$$

for $m \in \mathcal{M}, k \in \{1, 2, \cdots, K\}$. Since $z'_{m,k} \in B(g_m^{-1} z_{m,k}, \delta)$ one has by (4.1) and (4.2) that

$$\begin{aligned} d(h^{-1} x_{m,k}, h^{-1} x_{m,k'}) &= d(g_m z'_{m,k}, g_m z'_{m,k'}) \\ &\geq d(z_{m,k}, z_{m,k'}) - d(g_m z'_{m,k}, z_{m,k}) - d(g_m z'_{m,k'}, z_{m,k'}) \\ &> \epsilon \end{aligned}$$

for $m \in \mathcal{M}, 1 \leq k < k' \leq K$. Therefore, the conditions in Proposition 3.1 are valid for (X,G).

By Proposition 3.1, $h^*_{top}(X,G) \geq \log(1-\gamma)K$. Let $\gamma \searrow 0$. One has

$$h^*_{top}(X,G) \geq \log K.$$

This ends the proof of Theorem 1.4. □

4.2. **Proofs of Theorem 1.6 and Theorem 1.10.** We collect some known properties about almost finite-to-one extensions. Let $\pi : (X,G) \to (H,G)$ be an extension between two G-systems. Let $\pi^{-1} : H \to 2^X, y \to \pi^{-1}(y)$, where 2^X is the collection of nonempty closed subsets of X endowed with the Hausdorff metric d_H. Here the Hausdorff metric d_H is defined by

$$d_H(A,B) = \max\{\max_{a\in A} d(a,B), \max_{b\in B} d(b,A)\},$$

where $d(x,A) = \inf_{y\in A} d(x,y)$ for $x \in X$ and $A \in 2^X$. It is well known that $(2^X, d_H)$ is a compact metric space, as X is a compact metric space. We say a sequence $\{A_i\}_{i=1}^\infty$ in 2^X converges to $A \in 2^X$, denoted by $\lim_{i\to\infty} A_i = A$, if

$$\lim_{i\to\infty} d_H(A_i, A) = 0.$$

It follows from [26, 27] that π^{-1} is a upper semicontinuous map, and the set H_c of continuous points of π^{-1} is a dense G_δ subset of H. Let

$$H' = \mathrm{cl}(\{\pi^{-1}(y) : y \in H_c\}),$$

where the closure is taken in 2^X. Note that for each $A \in H'$, there is some $y \in H$ such that $A \subset \pi^{-1}(y)$, and hence $A \mapsto y$ define a map $\tau : H' \to H$. In [35], Veech showed that (H', G) is a minimal topological dynamical system and $\tau : (H',G) \to (H,G)$ is an almost one-to-one factor map. The following lemma is from [18, Corollary 2.19].

Lemma 4.2. *Let $\pi : (X,G) \to (H,G)$ be an extension with (H,G) being minimal. If π is an almost N-to-one extension for some $N \in \mathbb{N}$, then the cardinality of each element of H' is N, where (H',G) is the minimal system as defined above.*

Now we prove Theorem 1.6, which is motivated by [18, Theorem 3.1].

Proof of Theorem 1.6. Let (X,G) be a minimal G-system and $\pi : (X,G) \to (H,G)$ be the factor map on the maximal equicontinuous factor of (X,G). Let ν be the Haar measure of H. Let τ and (H',G) be defined as in above. Then τ is almost one-to-one and $|A| = K_2$ for $A \in H'$. The two maps τ^{-1} and π^{-1} are upper semicontinuous and hence measurable. By Lusin's theorem, we may therefore choose a compact set E_1 of H with $\nu(E_1) > 0$ such that $\tau^{-1}|_{E_1}$ and $\pi^{-1}|_{E_1}$ are continuous and $|\pi^{-1}(h)| = K_1$ for $h \in E_1$. Let E be the support of $\nu|_{E_1}$ and $h_0 \in E$. Then E is a compact subset of E_1 with $\nu(U \cap E) > 0$ for all open neighborhoods U of h_0.

Now we fix $K \in \mathbb{N}$ with $2 \leq K \leq \lceil \frac{K_1}{K_2} \rceil$ and $L = (K-1)K_2 + 1 \leq K_1$. Then there are distinct $x_1, x_2, \cdots, x_L \in \pi^{-1}(h_0)$. Since (X,G) is minimal, $\cup_{A\in\tau^{-1}(h_0)} A =$

$\pi^{-1}(h_0)$. We can find $L' \in \mathbb{N}$ and distinct $A_1, A_2, \cdots, A_{L'} \in \tau^{-1}(h_0)$ such that $\{x_1, x_2, \cdots, x_L\} \subset \bigcup_{l=1}^{L'} A_l$. Then by Lemma 4.2,

$$L' \geq \lceil \frac{L}{K_2} \rceil = K \geq 2.$$

We have the following claim.

Claim 4.3. *$(A_1, A_2, \cdots, A_{L'}) \in IT_{L'}(H', G)$.*

Proof of Claim 4.3. Put

$$r := \min_{\substack{x, x' \in \bigcup_{l=1}^{L'} A_l \\ x \neq x'}} d(x, x') > 0.$$

For each $l \in \{1, 2, \ldots, L'\}$, let if $A_l = \{x_1^l, x_2^l, \cdots, x_{K_2}^l\}$, and for $\epsilon \in (0, \frac{r}{3})$, define

$$U_l = \mathrm{cl}(\{\{y_1, y_2, \cdots, y_{K_2}\} \in H' : d(x_k, y_k) \leq \epsilon \text{ for } k \in \{1, 2, \cdots, K_2\}\}).$$

Clearly, each U_l is a closed neighborhood of A_l in H'. Therefore, to obtain that $(A_1, A_2, \cdots, A_{L'}) \in \mathrm{IT}_{L'}(H', G)$, by the arbitrariness of $\epsilon > 0$, it suffices to show that the tuple $(U_1, U_2, \cdots, U_{L'})$ has an infinite independence set.

Since $\epsilon \in (0, \frac{r}{3})$, $U_l \cap U_{l'} = \emptyset$ for $1 \leq l < l' \leq L'$. Since τ is almost one-to-one, it follows from [10, Lemma 2.4] that $\mathrm{cl}(\mathrm{int}(\tau(U_l))) = \tau(U_l)$ for $1 \leq l \leq L'$ and $\mathrm{int}(\tau(U_l)) \cap \mathrm{int}(\tau(U_{l'})) = \emptyset$ for $1 \leq l < l' \leq L'$. By the continuity of $\tau^{-1}|_E$, there is a neighborhood U of h_0 such that for $h \in U \cap E$, $\tau^{-1}(h) \cap U_l \neq \emptyset$ for $l \in \{1, 2, \cdots, L'\}$. One has $\nu(\bigcap_{l=1}^{L'} \tau(U_l)) \geq \nu(E \cap U) > 0$.

Let $E(H)$ be the Ellis semigroup of (H, G). By Theorem 2.5, (H, G) is a factor of $(E(H), G)$, where the open factor map π_e is given by $\pi_e : E(H) \to H, t \mapsto th$ for some fixed $h \in H$. Let $\tilde{\nu}$ be the Haar measure on $E(H)$. Then $\pi_e(\tilde{\nu}) = \nu$.

$$\begin{tikzcd}
& (X,G) \arrow[d, "\pi"'] & (H',G) \arrow[dl, "\tau"'] \\
(E(H),\tilde{\nu},G) \arrow[r, "\pi_e"] & (H,\nu,G) &
\end{tikzcd}$$

Put $V_l = \pi_e^{-1}(\tau(U_l))$ for $l \in \{1, 2, \cdots, L'\}$. Since π_e is open, one has $\mathrm{cl}(\mathrm{int}(V_l)) = V_l$ for $1 \leq l \leq L'$, $\mathrm{int}(V_l) \cap \mathrm{int}(V_{l'}) = \emptyset$ for $1 \leq l < l' \leq L'$ and $\tilde{\nu}(\bigcap_{l=1}^{L'} V_l) \geq \nu(\bigcap_{l=1}^{L'} \tau(U_l)) \geq \nu(E \cap U) > 0$. By Proposition 2.6, there exists an infinite set $\mathcal{A} = \{g_1, g_2, \cdots\} \subset G$ such that for all $a \in \{1, 2, \cdots, L'\}^{\mathbb{N}}$, $\bigcap_{n=1}^{\infty} g_n^{-1} \mathrm{int}(V_{a_n}) \neq \emptyset$. This implies for all $N \in \mathbb{N}$ and $a \in \{1, 2, \cdots, L'\}^N$, $\bigcap_{n=1}^{N} g_n^{-1} \mathrm{int}(\tau(U_{a_n}))$ is a nonempty open subset of H. Since τ is almost one-to-one, there is $h_{a,N} \in \bigcap_{n=1}^{N} g_n^{-1} \mathrm{int}(\tau(U_{a_n}))$ such that $|\tau^{-1}(h_{a,N})| = 1$. This implies $|\tau^{-1}(g_n h_{a,N})| = 1$ for $n = 1, 2, \cdots, N$ and then $\tau^{-1}(g_n h_{a,N}) \subset U_{a_n}$. Therefore, $\bigcap_{n=1}^{N} g_n^{-1} U_{a_n} \neq \emptyset$ for all $a \in \{1, 2, \cdots, L'\}^N$ and $N \in \mathbb{N}$. Since all U_l are compact, one has $\bigcap_{n=1}^{\infty} g_n^{-1} U_{a_n} \neq \emptyset$ for all $a \in \{1, 2, \cdots, L'\}^{\mathbb{N}}$. This implies the tuple $(U_1, U_2, \cdots, U_{L'})$ has an infinite independence set and then $(A_1, A_2, \cdots, A_{L'}) \in \mathrm{IT}_{L'}(H', G)$. $\square$

Let (X^{K_2}, G) be the product system and $M_{K_2}(X) = \{\{z_1, z_2, \cdots, z_{K_2}\} : z_i \in X, 1 \le i \le K_2\}$. Define the factor map

$$p : X^{K_2} \to M_{K_2}(X) : (z_1, \cdots, z_{K_2}) \to \{z_1, \cdots, z_{K_2}\}.$$

By Claim 4.3, $(A_1, A_2, \cdots, A_{L'}) \in \mathrm{IT}_{L'}(H', G) \subset \mathrm{IT}_{L'}(M_{K_2}(X), G)$. By Proposition 2.4 (4), there is $(A'_1, A'_2, \cdots, A'_{L'}) \in \mathrm{IT}_{L'}(X^{K_2}, G)$ with $(p(A'_1), p(A'_2), \cdots, p(A'_{L'})) = (A_1, A_2, \cdots, A_{L'})$. Then

$$\bigcup_{l=1}^{L'} p(A'_l) = \bigcup_{l=1}^{L'} A_l \supset \{x_1, x_2, \cdots, x_L\}.$$

For $k \in \{1, 2, \cdots, K_2\}$, let p_k be the projection on the k-th coordinate. Then there is $k_0 \in \{1, 2, \cdots, K_2\}$ such that

$$|\{p_{k_0}(A'_l) : 1 \le l \le L'\}| \ge \left\lceil \frac{|\bigcup_{l=1}^{L'} p(A'_l)|}{K_2} \right\rceil \ge \left\lceil \frac{L}{K_2} \right\rceil = K.$$

Let $z_1, z_2, \cdots, z_K$ be K distinct points in $\{p_{k_0}(A'_l) : 1 \le l \le L'\}$. Then one has $(z_1, z_2, \cdots, z_K) \in \mathrm{IT}^e_K(X, G)$. This ends the proof of Theorem 1.6. □

Now we prove Theorem 1.10.

Proof of Theorem 1.10. Let (X, G) be a G-system and $\pi : (X, G) \to (H, G)$ be the factor map on the maximal equicontinuous factor of (X, G). Let ν be the Haar measure of H. Assume that (X, G) satisfies the following:

(1) π is almost N-to-one for some $N \in \mathbb{N}$;
(2) (X, G) has no K-IT-tuples for some $K \in \mathbb{N}, K \ge 2$.

Assume that $|\mathcal{M}^e(X, G)| > N(K-1)$. There exist distinct $\mu_1, \cdots, \mu_{N(K-1)+1} \in \mathcal{M}^e(X, G)$. There exist disjoint measurable subsets $V_1, \cdots, V_{N(K-1)+1}$ of X such that

$$\mu_i(V_i) \ge 1 - \frac{1}{2(N(K-1)+1)} \text{ for } i = 1, 2, \cdots, N(K-1)+1.$$

Then

$$\nu(\cap_{i=1}^{N(K-1)+1} \pi(V_i)) \ge \frac{1}{2}.$$

This implies $|\pi^{-1}(h)| \ge N(K-1)+1$ for all $h \in \bigcap_{i=1}^{N(K-1)+1} \pi(V_i)$. By ergodicity, $\nu(\{h \in H : |\pi^{-1}(h)| \ge N(K-1)+1\}) = 1$. Then by Theorem 1.6, $\mathrm{IT}_K(X, G) \setminus \triangle^{(K)}(X) \neq \emptyset$ which contradicts the assumption (2). We finish the proof of Theorem 1.10. □

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Appendix A. Proof of Lemma 3.5

Let (X,μ,G) be a G-mps. Recall that

$$\mathcal{I}_\mu(G)=\{B\in\mathcal{B}_X^\mu:\mu(B\Delta(gB))=0,\ \text{for all } g\in G\},$$

and the Kronecker σ-algebra of (X,μ,G) is defined by

$$\mathcal{K}_\mu(G)=\{B\in\mathcal{B}_X^\mu:\{U_g1_B:g\in G\}\text{ is precompact in } L^2(\mu)\}.$$

In fact, $\mathcal{K}_\mu(G)$ is the σ-algebra generated by all almost periodic function $f\in L^2(\mu)$, i.e., $\{U_gf:g\in G\}$ is precompact in $L^2(\mu)$. We have the following.

Lemma A.1. *Let $(X_i,\mu_i,G),i=1,2$ be G-mpss. Then*

$$\mathcal{I}_{\mu_1\times\mu_2}(G)\subset\mathcal{K}_{\mu_1}(G)\times\mathcal{K}_{\mu_2}(G).$$

Proof. This proof is similar to that of [15, Theorem 16, pp. 53]. Let $F(x_1,x_2)\in L^2(\mu_1\times\mu_2)$ be G-invariant. Define a bounded operator $A:L^2(\mu_1)\to L^2(\mu_2)$ by setting

$$Ah(x_2)=\int_{X_1}F(x_1,x_2)h(x_1)d\mu_1(x_1)\text{ for } h\in L^2(\mu_1).$$

The operator A is a Hilbert-Schmidt operator and thus is compact. The operator A^*A is a positive, self-adjoint, compact operator on $L^2(\mu_1)$. Therefore, there exist a decreasing sequence $(t_j)_{j\in I}$ ($I=\{1,2,\ldots,N\}$ for some $N\in\mathbb{N}\cup\{\infty\}$) of positive numbers tending to 0 as $j\to\infty$ and an orthogonal sequence $(f_j)_{j\in I}$ in $L^2(\mu_1)$ such that for every $j\in I$,

$$A^*Af_j=t_jf_j,$$

and the linear subspace V is spanned by $(f_j)_{j\in I}$ is orthogonal to $\ker(A)$, that is, $A(V^\perp)=0$.

For all $t>0$, let $V_t=\operatorname{span}\{f_j:t_j=t,\ j\in I\}$. Since $F(x_1,x_2)$ is G-invariant, it follows that for all $g\in G$ and $h\in L^2(\mu_1)$

$$\begin{aligned}(U_gA(h))(x_2)&=\int_{X_1}F(x_1,gx_2)h(x_1)d\mu_1(x_1)\\&=\int_{X_1}F(gx_1,gx_2)h(gx_1)d\mu_1(x_1)\\&=\int_{X_1}F(x_1,x_2)h(gx_1)d\mu_1(x_1)\\&=A(U_gh)(x_2).\end{aligned}\tag{A.1}$$

Since each $U_g^* = U_{g^{-1}}$, one has

$$U_g A^* A = A^* A U_g, \text{ for all } g \in G.$$

Therefore, for each $t > 0$, V_t is G-invariant, and hence V is G-invariant. This means that for any $f \in V$

$$\{U_g f : g \in G\} \subset \{\tilde{f} \in V : \|\tilde{f}\|_2 = \|f\|_2\}.$$

Since for all $t > 0$, V_t are finite-dimensional linear spaces, $\{U_g f : g \in G\}$ is precompact in $L^2(\mu_1)$ and then $f \in L^2(X_1, \mathcal{K}_{\mu_1}(G), \mu_1)$. Denote $W = A(V)$. By (A.1), $W \subset L^2(X_2, \mathcal{K}_{\mu_2}(G), \mu_2)$ is a closed linear subspace. Let P_V, P_W and $P_{V\otimes W}$ be the projection on V, W and $V \otimes W$, respectively. Recall that $A(V^\perp) = 0$. Then for all $h_1 \in L^2(\mu_1), h_2 \in L^2(\mu_2)$, one has

$$\begin{aligned}
&\int_{X_1\times X_2} F(x_1,x_2)h_1(x_1)h_2(x_2)d\mu_1\times\mu_2(x_1,x_2)\\
&=\int_{X_2}(Ah_1)h_2 d\mu_2\\
&=\int_{X_2}(AP_V h_1)h_2 d\mu_2\\
&=\int_{X_2}(AP_V h_1)(P_W h_2)d\mu_2\\
&=\int_{X_1\times X_2} F(P_{V\otimes W}(h_1\otimes h_2))d\mu_1\times\mu_2\\
&=\int_{X_1\times X_2}(P_{V\otimes W}F)(h_1\otimes h_2)d\mu_1\times\mu_2,
\end{aligned}$$

which implies $F = P_{V\otimes W}F$. Since $V \otimes W \subset L^2(X_1 \times X_2, \mathcal{K}_{\mu_1\times\mu_2}(G), \mu_1 \times \mu_2)$, it follows that F is $\mathcal{K}_{\mu_1}(G) \times \mathcal{K}_{\mu_2}(G)$-measurable. Hence

$$\mathcal{I}_{\mu_1\times\mu_2}(G) \subset \mathcal{K}_{\mu_1} \times \mathcal{K}_{\mu_2}.$$

This ends the proof of Lemma A.1. □

Now we prove Lemma 3.5.

Proof. Let $I \in \mathbb{N}$ and $(X_i, \mu_i, G), i = 1, 2, \cdots, I$ be G-mpss. If $I = 1$, it is clear true. Now we suppose that $I \geq 2$. For all $i \in \{1, 2, \cdots, I\}$, define a map $\pi_i : X_1 \times \cdots X_{i-1} \times X_{i+1} \cdots \times X_I \times X_i \to X_1 \times \cdots \times X_I, (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_I, x_i) \mapsto (x_1, \cdots, x_I)$. By Lemma A.1, for all $i \in \{1, 2, \cdots, I\}$,

$$\begin{aligned}
&\mathcal{I}_{\mu_1\times\mu_2\times\cdots\times\mu_I}\\
&=\pi_i(\mathcal{I}_{(\mu_1\times\mu_2\times\cdots\times\mu_{i-1}\times\mu_{i+1}\cdots\times\mu_I)\times\mu_i})\\
&\subset\pi_i(\mathcal{K}_{\mu_1\times\mu_2\times\cdots\times\mu_{i-1}\times\mu_{i+1}\cdots\times\mu_I}(G)\times\mathcal{K}_{\mu_i}(G))\\
&\subset\pi_i(\mathcal{B}_X^{\mu_1}\times\mathcal{B}_X^{\mu_2}\times\cdots\times\mathcal{B}_X^{\mu_{i-1}}\times\mathcal{B}_X^{\mu_{i+1}}\times\mathcal{B}_X^{\mu_{i+2}}\times\cdots\times\mathcal{B}_X^{\mu_I}\times\mathcal{K}_{\mu_i}(G))\\
&=\mathcal{B}_X^{\mu_1}\times\mathcal{B}_X^{\mu_2}\times\cdots\times\mathcal{B}_X^{\mu_{i-1}}\times\mathcal{K}_{\mu_i}(G)\times\mathcal{B}_X^{\mu_{i+1}}\times\mathcal{B}_X^{\mu_{i+2}}\times\cdots\times\mathcal{B}_X^{\mu_I}.
\end{aligned}$$

This implies $\mathcal{I}_{\mu_1\times\mu_2\times\cdots\times\mu_I}(G)\subset\mathcal{K}_{\mu_1}(G)\times\mathcal{K}_{\mu_2}(G)\times\cdots\times\mathcal{K}_{\mu_I}(G)$. We finish the proof of Lemma 3.5. $\square$

Appendix B. Proof of Theorem 2.9

Firstly we show that $h^*_\mu(G,\alpha)=H_\mu(\alpha|\mathcal{K}_\mu)$ for any G-mps (X,μ,G) and $\alpha\in\mathcal{P}^\mu_X$, where $\mathcal{K}_\mu=\mathcal{K}_\mu(G)$ is the Kronecker σ-algebra of (X,μ,G). The following lemma is from [36, pp. 94].

Lemma B.1. *Let $(X,\mathcal{B},\mu)$ be a Borel probability space and $r\geq 1$ be a fixed integer. For each $\epsilon>0$, there exists $\delta=\delta(\epsilon,r)>0$, such that if $\alpha=\{A_1,A_2,\ldots,A_r\}$ and $\eta=\{B_1,B_2,\ldots,B_r\}$ are any two finite measurable partitions of $(X,\mathcal{B},\mu)$ with $\sum_{j=1}^r\mu(A_j\Delta B_j)<\delta$ then $H_\mu(\alpha|\eta)+H_\mu(\eta|\alpha)<\epsilon$.*

By Lemma B.1, we can get the following result.

Lemma B.2. *Let (X,μ,G) be a G-mps and $B\in\mathcal{B}^\mu_X$. If $\{U_g1_B:g\in G\}$ is precompact in $L^2(\mu)$, then $h^*_\mu(G,\{B,B^c\})=0$.*

Proof. Let $\eta=\{B,B^c\}$. If $\{U_g1_B:g\in G\}$ is precompact in $L^2(\mu)$, then for any infinite sequence $\mathcal{A}=\{g_i\}_{i\in\mathbb{N}}$ of G, $\{U_{g_i}1_B:i\in\mathbb{N}\}$ is precompact in $L^2(\mu)$. So for $\epsilon>0$, there exists $s\in\mathbb{N}$ such that for any $i\in\mathbb{N}$ there is $j_i\in\{1,2,\cdots,s\}$ such that

$$\mu(g_i^{-1}B^c\Delta g_{j_i}^{-1}B^c)=\mu(g_i^{-1}B\Delta g_{j_i}^{-1}B)=\|U_{g_i}1_B-U_{g_{j_i}}1_B\|_2^2<\frac{\delta}{2}.$$

By Lemma B.1, one has

$$H_\mu(g_i^{-1}\eta|g_{j_i}^{-1}\eta)+H_\mu(g_{j_i}^{-1}\eta|g_i^{-1}\eta)<\epsilon \text{ for } i\in\mathbb{N}.$$

Thus for any $i>s$, we have

$$H_\mu(g_i^{-1}\eta|\bigvee_{j=1}^{i-1}g_j^{-1}\eta)\leq H_\mu(g_i^{-1}\eta|g_{j_i}^{-1}\eta)<\epsilon.$$

Hence

$$h^{\mathcal{A}}_\mu(G,\eta)=\limsup_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}H_\mu(g_i^{-1}\eta|\bigvee_{j=1}^{i-1}g_j^{-1}\eta)\leq\epsilon.$$

Let $\epsilon\to 0$. We obtain that $h^{\mathcal{A}}_\mu(G,\eta)=0$. By the arbitrariness of $\mathcal{A}$, we have $h^*_\mu(G,\eta)=0$. This ends the proof of Lemma B.2. $\square$

Theorem B.3. *Let (X,μ,G) be a G-mps, and $\alpha\in\mathcal{P}^\mu_X$. Then $h^*_\mu(G,\alpha)=H_\mu(\alpha|\mathcal{K}_\mu)$.*

Proof. We firstly, show that $h^*_\mu(G,\alpha)\leq H_\mu(\alpha|\mathcal{K}_\mu)$. Let $\mathcal{A}=\{g_1,g_2,\cdots\}$ be a sequence of G. Since $(X,\mathcal{B}^\mu_X,\mu)$ is separable, there exist countably many finite measurable partitions $\{\eta_k\}_{k\in\mathbb{N}}\subset\mathcal{K}_\mu$, such that $\lim\limits_{k\to\infty}H_\mu(\alpha|\eta_k)=H_\mu(\alpha|\mathcal{K}_\mu)$. Hence for any $k \in \mathbb{N}$, we have

$$h_\mu^{\mathcal{A}}(G,\alpha) = \limsup_{l\to\infty} \frac{1}{l} H_\mu\big(\bigvee_{i=1}^{l} g_i^{-1}\alpha\big) \le \limsup_{l\to\infty} \frac{1}{l} H_\mu(\bigvee_{i=1}^{l} g_i^{-1}(\alpha \vee \eta_k))$$

Since $\eta_k \subset \mathcal{K}_\mu$ for all $k \in \mathbb{N}$, one has by Lemma B.2 that

$$h_\mu^{\mathcal{A}}(G,\eta_k) = \lim_{l\to\infty} \frac{1}{l} H_\mu(\bigvee_{i=1}^{l} g_i^{-1}\eta_k) = 0.$$

Hence

$$\begin{aligned}
h_\mu^{\mathcal{A}}(G,\alpha) &\le \limsup_{l\to\infty} \frac{1}{l} H_\mu(\bigvee_{i=1}^{l} g_i^{-1}(\alpha \vee \eta_k)) - \lim_{l\to\infty} \frac{1}{l} H_\mu(\bigvee_{i=1}^{l} g_i^{-1}\eta_k) \\
&= \limsup_{l\to\infty} \frac{1}{l} H_\mu\big(\bigvee_{i=1}^{l} g_i^{-1}\alpha | \bigvee_{i=1}^{l} g_i^{-1}\eta_k\big) \\
&\le \limsup_{l\to\infty} \frac{1}{l} \sum_{i=1}^{l} H_\mu(g_i^{-1}\alpha | g_i^{-1}\eta_k) \\
&= H_\mu(\alpha|\eta_k).
\end{aligned}$$

Finally, let $k \to \infty$. We get $h_\mu^{\mathcal{A}}(G,\alpha) \le H_\mu(\alpha|\mathcal{K}_\mu)$. By the arbitrariness of $\mathcal{A}$, we have $h_\mu^*(G,\alpha) \le H_\mu(\alpha|\mathcal{K}_\mu)$.

Now we prove the opposite side. Given any $\beta \in \mathcal{P}_X^\mu$, write $\alpha = \{A_1, A_2, \dots, A_l\}$ and $\beta = \{B_1, B_2, \dots, B_t\}$. By Theorem 2.8, $1_{A_i} - \mathbb{E}(1_{A_i}|\mathcal{K}_\mu) \in \mathcal{H}_{wm}$ for $1 \le i \le l$. There exists a sequence $\mathcal{A} = \{g_i\}_{i=1}^\infty$ of G such that

$$\lim_{i\to\infty} \langle U_{g_i}(1_{A_k} - \mathbb{E}(1_{A_k}|\mathcal{K}_\mu)), 1_{B_j}\rangle = 0 \tag{B.1}$$

for any $1 \le k \le l$ and $1 \le j \le t$. Hence

$$\begin{aligned}
&\liminf_{i\to\infty} H_\mu(g_i^{-1}\alpha|\beta) \\
=& \liminf_{i\to\infty} \sum_{k,j} -\mu\left(g_i^{-1}A_k \cap B_j\right) \log\left(\frac{\mu(g_i^{-1}A_k \cap B_j)}{\mu(B_j)}\right) \\
=& \liminf_{i\to\infty} \sum_{k,j} -\langle U_{g_i}1_{A_k}, 1_{B_j}\rangle \log\left(\frac{\langle U_{g_i}1_{A_k}, 1_{B_j}\rangle}{\mu\left(B_j\right)}\right) \\
\overset{(B.1)}{=}& \liminf_{i\to\infty} \sum_{k,j} -\langle U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right), 1_{B_j}\rangle \log\left(\frac{\langle U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right), 1_{B_j}\rangle}{\mu\left(B_j\right)}\right).
\end{aligned}$$

Let

$$a_{kj}^i = -\langle U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right), 1_{B_j}\rangle \log\left(\frac{\langle U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right), 1_{B_j}\rangle}{\mu\left(B_j\right)}\right) \text{ and } \mu_{B_j}(\cdot) = \frac{\mu(\cdot \cap B_j)}{\mu(B_j)}.$$

By the concavity of $-x\log x$, we conclude that

$$\begin{aligned}\frac{a_{kj}^i}{\mu(B_j)} &= -\left(\int_{B_j}\frac{U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)}{\mu\left(B_j\right)}\,\mathrm{d}\mu\right)\log\left(\int_{B_j}\frac{U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)}{\mu\left(B_j\right)}\,\mathrm{d}\mu\right)\\ &= -\left(\int_{B_j}U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)\mathrm{d}\mu_{B_j}\right)\log\left(\int_{B_j}U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)\mathrm{d}\mu_{B_j}\right)\\ &\geq -\int_{B_j}U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)\log\left(U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)\right)\mathrm{d}\mu_{B_j}\\ &= -\int_{B_j}\frac{U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)}{\mu\left(B_j\right)}\log\left(U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)\right)\mathrm{d}\mu.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\sum_{k,j}a_{kj}^i &\geq \sum_{k,j}-\int_{B_j}U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)\log\left(U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)\right)\mathrm{d}\mu\\ &= \sum_{k}-\int_{X}U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)\log\left(U_{g_i}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)\right)\mathrm{d}\mu\\ &= \sum_{k}-\int_{X}\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)\log\left(\mathbb{E}\left(1_{A_k}|\mathcal{K}_\mu\right)\right)\mathrm{d}\mu\\ &= H_\mu(\alpha|\mathcal{K}_\mu).\end{aligned}$$

This shows that

$$\liminf_{i\to\infty}H_\mu\left(g_i^{-1}\alpha|\beta\right)\geq H_\mu\left(\alpha|\mathcal{K}_\mu\right). \tag{B.2}$$

By the above discussion, we can obtain inductively a sequence $\mathcal{A}'=\{h_i\}_{i=1}^\infty$ of G such that for each $i\in\mathbb{N}$, one has

$$H_\mu\left(h_i^{-1}\alpha|\bigvee_{j=1}^{i-1}h_j^{-1}\alpha\right)\geq H_\mu(\alpha|\mathcal{K}_\mu)-\frac{1}{2^i}.$$

Thus,

$$\begin{aligned}H_\mu\left(\bigvee_{i=1}^n h_i^{-1}\alpha\right) &= H_\mu(h_1^{-1}\alpha)+H_\mu(h_2^{-1}\alpha|h_1^{-1}\alpha)+\ldots+H_\mu\left(h_n^{-1}\alpha|\bigvee_{j=1}^{n-1}h_j^{-1}\alpha\right)\\ &\geq \sum_{i=1}^n\left(H_\mu(\alpha|\mathcal{K}_\mu)-\frac{1}{2^i}\right)\geq n\cdot H_\mu(\alpha|\mathcal{K}_\mu)-1.\end{aligned}$$

Therefore, we have

$$\begin{aligned}h_\mu^{\mathcal{A}'}(G,\alpha) &= \limsup_{n\to\infty}\frac{1}{n}H_\mu\left(\bigvee_{i=1}^n h_i^{-1}\alpha\right)\\ &\geq \limsup_{n\to\infty}\frac{nH_\mu(\alpha|\mathcal{K}_\mu)-1}{n}=H_\mu(\alpha|\mathcal{K}_\mu).\end{aligned}$$

This implies $h^*_\mu(G,\alpha) \geq h^{\mathcal{A}'}_\mu(G,\alpha) \geq H_\mu(\alpha|\mathcal{K}_\mu)$.

Therefore $h^*_\mu(G,\alpha) = H_\mu(\alpha|\mathcal{K}_\mu)$. This ends the proof of Theorem B.3. □

We need the following lemma from [33, Lemma 3 in §4 No. 2].

Lemma B.4. *Let μ be a probability measure on X and $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{B}^\mu_X$. Let $\mu = \int \mu_y d\mu(y)$ be the disintegration of μ over $\mathcal{F}$. Suppose μ_y is non-atomic for μ-a.e. $y \in X$. If $0 \leq r \leq 1$ and $A \in \mathcal{B}^\mu_X$ with $\mu_y(A) \leq r$ for μ-a.e. $y \in X$, then there exists $A' \in \mathcal{B}^\mu_X$ such that $A \subset A'$ and $\mu_y(A') = r$ for μ-a.e. $y \in X$.*

Now we prove Theorem 2.9.

Proof of Theorem 2.9. Let (X,μ,G) be an ergodic G-mps and $\mu = \int_X \mu_x d\mu(x)$ be the disintegration of μ over $\mathcal{K}_\mu(G)$.

We prove $h^*_\mu(X,G) \leq \log k_\mu$ firstly. If $k_\mu = \infty$, this is clear true. Assume that $k_\mu \in \mathbb{N}$. Then for $\alpha \in \mathcal{P}^\mu_X$,

$$|\{A \in \alpha : \mu_x(A) > 0\}| \leq k_\mu \text{ for } \mu\text{-a.e. } x \in X.$$

Hence by the convex of $-x\log x$ one has

$$\begin{aligned} H_\mu(\alpha|\mathcal{K}_\mu) &= \int_X \sum_{A\in\alpha} -\mathbb{E}_\mu(1_A|\mathcal{K}_\mu)(x)\log \mathbb{E}_\mu(1_A|\mathcal{K}_\mu)(x)d\mu(x) \\ &= \int_X \sum_{A\in\alpha} -\mu_x(A)\log\mu_x(A)d\mu(x) \\ &= \int_X \sum_{\substack{A\in\alpha \\ \mu_x(A)>0}} -\mu_x(A)\log\mu_x(A)d\mu(x) \\ &\leq \int_X -(\sum_{\substack{A\in\alpha \\ \mu_x(A)>0}} \mu_x(A))\log\frac{\sum_{\substack{A\in\alpha \\ \mu_x(A)>0}}\mu_x(A)}{|\{A\in\alpha:\mu_x(A)>0\}|}d\mu(x) \\ &\leq \int_X \log k_\mu d\mu(x) = \log k_\mu. \end{aligned}$$

By Theorem B.3, we have $h^*_\mu(G,\alpha) = H_\mu(\alpha|\mathcal{K}_\mu) \leq \log k_\mu$. With the arbitrariness of α, we have $h^*_\mu(X,G) \leq \log k_\mu$.

Now we prove $h^*_\mu(X,G) \geq \log k_\mu$. If $k_\mu = \infty$, since (X,μ,G) is ergodic, μ_x is atomless for μ-a.e. $x \in X$. Then by Lemma B.4, for $k \in \mathbb{N}$ there exists $\alpha \in \mathcal{P}^\mu_X$ depending on k such that

$$\mu_x(A) = \frac{1}{k} \text{ for } \mu\text{-a.e.} x \in X \text{ and } A \in \alpha.$$

Then $h^*_\mu(G) \geq h^*_\mu(G,\alpha) = H_\mu(\alpha|\mathcal{K}_\mu) = \log k$. Letting $k \to \infty$, one has $h^*_\mu(G) = \infty$.

If $k_\mu < \infty$, since (X,μ,G) is ergodic, $\mu_x(y) = \frac{1}{k_\mu}$ for $y \in \mathrm{supp}(\mu_x)$ for μ-a.e. $x \in X$. Since $x \to \mu_x$ is measurable, there is a measurable partition $\alpha = \{A_1, \cdots, A_{k_\mu}\}$

of X such that
$$\mu_x(A) = \frac{1}{k_\mu} \text{ for } \mu\text{-a.e. } x \in X \text{ and } A \in \alpha.$$
Then $h^*_\mu(G) \geq h^*_\mu(G, \alpha) = H_\mu(\alpha|\mathcal{K}_\mu) = \log k_\mu$.

Therefore $h^*_\mu(G) = \log k_\mu$. This ends the proof of Theorem 2.9. $\square$

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