

MEAN LI-YORKE CHAOS ALONG ANY INFINITE SEQUENCE FOR INFINITE-DIMENSIONAL RANDOM DYNAMICAL SYSTEMS

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ABSTRACT. In this paper, we study the mean Li-Yorke chaotic phenomenon along any infinite positive integer sequence for infinite-dimensional random dynamical systems. To be precise, we prove that if an injective continuous infinite-dimensional random dynamical system (X, ϕ) over an invertible ergodic Polish system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ admits a ϕ -invariant random compact set K with positive topological entropy, then given a positive integer sequence $\mathbf{a} = \{a_i\}_{i \in \mathbb{N}}$ with $\lim_{i \rightarrow +\infty} a_i = +\infty$, for \mathbb{P} -a.s. $\omega \in \Omega$ there exists an uncountable subset $S(\omega) \subset K(\omega)$ and $\epsilon(\omega) > 0$ such that for any two distinct points $x_1, x_2 \in S(\omega)$, one has

$$\liminf_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N d(\phi(a_i, \omega)x_1, \phi(a_i, \omega)x_2) = 0, \quad \limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N d(\phi(a_i, \omega)x_1, \phi(a_i, \omega)x_2) > \epsilon(\omega),$$

where d is a compatible complete metric on X . To overcome the difficulties arising in the infinite-dimensional random dynamical systems, we use the regularity of measure to construct the expected measurable partitions and to decompose the distance function.

1. INTRODUCTION

In the study of dynamical systems, there is a fundamental question: how to describe the chaotic phenomenon. In the past few decades, there are various results about this question, such as [1, 3, 10, 15, 20, 21, 30, 37, 38]. The reader can refer to [29] for more aspects and details. In this paper, we focus on the relation between the positive entropy and the mean Li-Yorke chaos in infinite-dimensional random dynamical systems. In deterministic finite-dimensional dynamical systems, Blanchard et al. [5] proved that the positive entropy implies the Li-Yorke chaos; moreover, Downarowicz [12] observed that the mean Li-Yorke chaos is equivalent to the DC2 chaos and proved that the positive entropy implies the mean Li-Yorke chaos. For further studying on the Li-Yorke chaos, see [4, 17, 28, 31, 41, 42].

Since about 1980, random dynamical systems have attracted widespread attention. There is an extensive literature on random dynamical systems, for example [2, 23, 25, 26, 27, 32, 36]. However, little is known about the relation between the entropy and the Li-Yorke chaos for random dynamical systems. Until 2017, the first result was obtained by Huang and Lu [19]. They showed that the positive entropy implies the Li-Yorke chaos for almost sure fibers in infinite-dimensional random dynamical systems. Recently, the second author [39] proved that the positive entropy implies the DC2 chaos for almost sure fibers. If the phase space is not compact, even though trajectories of two points are very close for most of the time, then the average of trajectories of these two points is not necessarily small, since trajectories of these two points may be pretty far for the rest time. Therefore, the DC2 chaos may not imply the mean Li-Yorke chaos in infinite-dimensional random dynamical systems which is different from the case in deterministic finite-dimensional dynamical systems.

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It is natural to ask that whether the positive entropy implies the mean Li-Yorke chaos in infinite-dimensional random dynamical systems. In this paper, we give a positive answer. The main difficulty in our proofs is that the phase space is not compact, even not locally compact. This leads to the invalidation of many classical results in compact systems. By a detailed observation, we show that there also are many asymptotic pairs in non-compact systems (see Theorem 3.1). Besides, we prove that the set of the pairs being separate in the mean sense along a given infinite positive integer sequence has the full measure under the product of conditional measure with respect to the relative Pinsker factor in non-compact systems with positive entropy (see Theorem 3.8). Through these two theorems, we show that the positive entropy implies the mean Li-Yorke chaos for almost sure fibers in infinite-dimensional random dynamical systems.

To describe the results of this paper precisely, we present the basic setting. Assume that (X, d) is a complete separable metric space and $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is an invertible ergodic Polish system (see Section 2.1 for the definition). Consider the discrete random dynamical system over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ defined by the measurable map

$$\phi : \mathbb{N}_0 \times \Omega \times X \rightarrow X, \quad (n, \omega, x) \mapsto \phi(n, \omega, x)$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, such that the map $\phi(n, \omega) := \phi(n, \omega, \cdot)$ for any $\omega \in \Omega$ satisfies the following properties:

- (i) $\phi(0, \omega) = \text{Id}_X$;
- (ii) $\phi(n + m, \omega) = \phi(n, \theta^m \omega) \circ \phi(m, \omega)$ for any $n, m \in \mathbb{N}_0$.

The pair (X, ϕ) is called a *continuous (resp. an injective continuous) random dynamical system* over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ if $\phi(n, \omega)$ is a continuous (resp. an injective continuous) from X to itself for any $n \in \mathbb{N}$ and \mathbb{P} -a.s. $\omega \in \Omega$.

A multifunction $K = \{K(\omega)\}_{\omega \in \Omega}$ is called a *random closed (resp. compact) set* if \mathbb{P} -a.s. $\omega \in \Omega$, $K(\omega)$ is a nonempty closed (resp. compact) set of X and

$$\omega \mapsto \inf_{y \in K(\omega)} d(x, y)$$

is measurable for any $x \in X$. In this paper, K will also be regarded as a subset of $\Omega \times X$ by $\{(\omega, x) : x \in K(\omega)\}$. A random closed set K is said to be ϕ -invariant if $\phi(n, \omega)K(\omega) = K(\theta^n \omega)$ for any $n \in \mathbb{N}$ and $\omega \in \Omega$.

The first main result of this paper is stated as follows.

Theorem 1.1. *Let (X, ϕ) be a continuous random dynamical systems over an invertible ergodic Polish system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and $K \subset \Omega \times X$ be a ϕ -invariant random compact set. If the topological entropy of (K, ϕ) is positive, i.e. $h_{\text{top}}(\phi, K) > 0$ (see Section 2.2), then (K, ϕ) is mean Li-Yorke chaotic for almost sure fibers. Namely, for \mathbb{P} -a.s. $\omega \in \Omega$ there exists a Mycielski set (i.e., a union of countably many Cantor sets) $S(\omega) \subset K(\omega)$ and $\epsilon(\omega) > 0$ such that for any distinct points $x_1, x_2 \in S(\omega)$, one has*

$$\liminf_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N d(\phi(i, \omega)x_1, \phi(i, \omega)x_2) = 0,$$

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N d(\phi(i, \omega)x_1, \phi(i, \omega)x_2) > \epsilon(\omega).$$

Furthermore, the Li-Yorke chaos along a given sequence has been investigated in the deterministic dynamical systems. For example, Huang, Li and Ye [18] showed that the positive entropy implies the chaos along any infinite sequence, and the reader can see [28, 31] for more results. Inspired by the above works, we prove that an injective continuous infinite-dimensional random

dynamical systems with positive entropy is mean Li-Yorke chaotic along any infinite sequence for almost sure fibers. Specifically,

Theorem 1.2. *Let (X, ϕ) be an injective continuous random dynamical system over an invertible ergodic Polish system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and $K \subset \Omega \times X$ be a ϕ -invariant random compact set. If the topological entropy of (K, ϕ) is positive, i.e. $h_{\text{top}}(\phi, K) > 0$, then given an infinite positive integer sequence $\mathbf{a} = \{a_i\}_{i \in \mathbb{N}}$ (i.e. $\lim_{i \rightarrow +\infty} a_i = +\infty$), (K, ϕ) is mean Li-Yorke chaotic along \mathbf{a} for almost sure fibers. Namely, for \mathbb{P} -a.s. $\omega \in \Omega$ there exists a Mycielski set $S(\omega) \subset K(\omega)$ and $\epsilon(\omega) > 0$ such that for any distinct points $x_1, x_2 \in S(\omega)$, one has*

$$(1.1) \quad \liminf_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N d(\phi(a_i, \omega)x_1, \phi(a_i, \omega)x_2) = 0,$$

$$(1.2) \quad \limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N d(\phi(a_i, \omega)x_1, \phi(a_i, \omega)x_2) > \epsilon(\omega).$$

This paper is organized as follows. In Section 2, we review some necessary notions and required properties. In Section 3, we prove that stable sets in a Polish system are dense in the support of conditional measure with respect to relative Pinsker factor, namely Theorem 3.1; at last, we prove the set of all separating pairs in a Polish system with positive entropy is full measure of the product of conditional measure with respect to the relative Pinsker factor, namely Theorem 3.8. In Section 4, we prove Theorem 1.1 and Theorem 1.2.

2. PRELIMINARY

In this section, we review some basic concepts and results from the theory of measure-preserving dynamical systems.

2.1. Conditional measure-theoretic entropy. In this subsection, we review the definition of conditional measure-theoretic entropy of a measure-preserving dynamical system and state some properties of the conditional measure-theoretic entropy. The reader can refer to [14, 40] for more details.

A Polish space X means that it is a separable topological space whose topology is metrizable by a complete metric. A Polish probability space (X, \mathcal{X}, μ) means that X is a Polish space, \mathcal{X} is the Borel σ -algebra, and μ is a probability measure on \mathcal{X} . In this moment, (X, \mathcal{X}) is called Polish measurable space. A probability space (X, \mathcal{X}, μ) is called a standard probability space if (X, \mathcal{X}) is isomorphic to a Polish measurable space. A measure-preserving dynamical system (MDS for short) (X, \mathcal{X}, μ, T) means that T is a measure-preserving map on the probability space (X, \mathcal{X}, μ) . A MDS (X, \mathcal{X}, μ, T) is called ergodic if for any $A \in \mathcal{X}$, $T^{-1}A = A$ implies that $\mu(A)\mu(X \setminus A) = 0$.

Given two MDSs (X, \mathcal{X}, μ, T) and (Y, \mathcal{Y}, ν, S) , we say that (Y, \mathcal{Y}, ν, S) is a factor of (X, \mathcal{X}, μ, T) if there exists a measure-preserving map $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ such that $\pi \circ T = S \circ \pi$, and π is called a factor map.

Definition 2.1. A MDS (X, \mathcal{X}, μ, T) is called a Polish system if (X, \mathcal{X}, μ) is a Polish probability space. If $T^{-1} : X \rightarrow X$ exists and is measurable, then (X, \mathcal{X}, μ, T) is called an invertible MDS.

Let (X, \mathcal{X}, μ) be a probability space and \mathcal{S} be a sub- σ -algebra of \mathcal{X} . The conditional expectation is a linear operator $\mathbb{E}(\cdot|\mathcal{S}) : \mathcal{L}^1(X, \mathcal{X}, \mu) \rightarrow \mathcal{L}^1(X, \mathcal{S}, \mu)$ characterized by following properties:

- (i) for every $f \in \mathcal{L}^1(X, \mathcal{X}, \mu)$, $\mathbb{E}(f|\mathcal{S})$ is \mathcal{S} -measurable;
- (ii) for any $A \in \mathcal{S}$ and $f \in \mathcal{L}^1(X, \mathcal{X}, \mu)$, $\int_A f d\mu = \int_A \mathbb{E}(f|\mathcal{S}) d\mu$.

Recall the Martingale theorem. The reader can see it in [16, Theorem 14.26] or [13, Chapter 5.2].

Theorem 2.2 (Martingale theorem). *Given a probability space (X, \mathcal{X}, μ) , suppose that $\{\mathcal{S}_n\}_{n \in \mathbb{N}}$ is a decreasing sequence (resp. an increasing sequence) of sub- σ -algebra of \mathcal{X} and $\mathcal{S} = \bigcap_{n \geq 1} \mathcal{S}_n$ (resp. $\mathcal{S} = \sigma(\bigvee_{n \geq 1} \mathcal{S}_n)$). Then for any $f \in \mathcal{L}^1(X, \mathcal{X}, \mu)$, $\mathbb{E}(f|\mathcal{S}_n) \rightarrow \mathbb{E}(f|\mathcal{S})$ as $n \rightarrow +\infty$ in $\mathcal{L}^1(X, \mathcal{X}, \mu)$ and μ -almost sure.*

Let (X, \mathcal{X}, μ, T) be a MDS. Given a finite measurable partition α and a T -invariant sub- σ -algebra \mathcal{S} of \mathcal{X} , denote

$$H_\mu(\alpha|\mathcal{S}) := \sum_{A \in \alpha} \int_X -\mathbb{E}(1_A|\mathcal{S}) \log \mathbb{E}(1_A|\mathcal{S}) d\mu,$$

Note that $\{H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha|\mathcal{S})\}_{n \in \mathbb{N}_0}$ is a non-negative and sub-additive sequence. Therefore, the conditional measure-theoretic entropy of μ with respect to \mathcal{S} is defined as

$$h_\mu(T|\mathcal{S}) := \sup_{\alpha} h_\mu(T, \alpha|\mathcal{S}) := \sup_{\alpha} \lim_{n \rightarrow +\infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} T^{-i}\alpha|\mathcal{S} \right),$$

where α runs over all finite measurable partitions of (X, \mathcal{X}) . Let $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ be a factor map between two MDSs. The conditional measure-theoretic entropy of μ with respect to π is defined as $h_\mu(T|\pi) := h_\mu(T|\pi^{-1}\mathcal{Y})$. The following result is a generalization of Abramov-Rohlin formula from [7].

Lemma 2.3. Let $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ and $\psi : (Y, \mathcal{Y}, \nu, S) \rightarrow (Z, \mathcal{Z}, \eta, R)$ be two factor maps between two MDSs on standard probability spaces. Then $\psi \circ \pi : (X, \mathcal{X}, \mu, T) \rightarrow (Z, \mathcal{Z}, \eta, R)$ is also a factor map and

$$h_\mu(T|\psi \circ \pi) = h_\mu(T|\pi) + h_\nu(S|\psi).$$

2.2. Entropy for random dynamical systems. In this subsection, we mainly introduce the entropy and the variational principle in random dynamical systems. Throughout this subsection, we assume that (X, ϕ) is a continuous random dynamical system over an ergodic Polish system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, where (X, d) is a complete separable metric space. The reader can refer to [2, 25] for more details.

Definition 2.4. Suppose that (X, ϕ) is a continuous random dynamical system over an invertible ergodic Polish system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. The map

$$\Phi : \Omega \times X \rightarrow \Omega \times X, \quad (\omega, x) \mapsto (\theta\omega, \phi(1, \omega)x)$$

is said to be a *skew product system* induced by (X, ϕ) .

Let $\pi_\Omega : \Omega \times X \rightarrow \Omega$ be the projection. A probability measure μ on the measurable space $(\Omega \times X, \mathcal{F} \times \mathcal{X})$ is said to have *marginal* \mathbb{P} if $(\pi_\Omega)_*\mu = \mathbb{P}$, namely $\mu(A \times X) = \mathbb{P}(A)$ for any measurable subset $A \in \mathcal{F}$. Denote $\mathcal{P}_\mathbb{P}(\Omega \times X)$ as the collection of such measures, $\mathcal{M}_\mathbb{P}(\Omega \times X, \Phi)$ as the collection of Φ -invariant elements of $\mathcal{P}_\mathbb{P}(\Omega \times X)$, and $\mathcal{E}_\mathbb{P}(\Omega \times X, \Phi)$ as the collection of ergodic elements of $\mathcal{M}_\mathbb{P}(\Omega \times X, \Phi)$. For convenience, we omit Φ and write $\mathcal{M}_\mathbb{P}(\Omega \times X, \Phi)$ and $\mathcal{E}_\mathbb{P}(\Omega \times X, \Phi)$ as $\mathcal{M}_\mathbb{P}(\Omega \times X)$ and $\mathcal{E}_\mathbb{P}(\Omega \times X)$, respectively.

Assume that K is a ϕ -invariant random compact set. Then there exists $\mu \in \mathcal{M}_\mathbb{P}(\Omega \times X)$ with $\mu(K) = 1$ (see [8] or [2, Theorem 1.6.13]). Set

$$\mathcal{M}_\mathbb{P}^K(\Omega \times X) = \{\mu \in \mathcal{M}_\mathbb{P}(\Omega \times X) : \mu(K) = 1\}.$$

Since K is a Borel subset of $\Omega \times X$ (see [2, Proposition 1.6.2]), $(K, \mathcal{K}, \mu, \Phi)$ is a MDS on a standard probability space where $\mathcal{K} = \{A \cap K : A \in \mathcal{F} \times \mathcal{X}\}$ and

$$\pi_\Omega : (K, \mathcal{K}, \mu, \Phi) \rightarrow (\Omega, \mathcal{F}, \mathbb{P}, \theta)$$

is a factor map between MDSs on standard probability spaces. The *measure-theoretic entropy* (K, ϕ) with respect to μ is defined by

$$h_\mu(\phi, K) := h_\mu(\Phi|_{\pi_\Omega}) = h_\mu(\Phi|_{\pi_\Omega^{-1}\mathcal{F}}).$$

That is, $h_\mu(\phi, K)$ is the conditional measure-theoretic entropy of $(K, \mathcal{K}, \mu, \Phi)$ with respect to π_Ω .

Now, we prepare to give the definition of the topological entropy of (K, ϕ) . The reader should be careful that the definition of the topological entropy here is slightly different from the definition of the topological entropy when X is compact ([6, 24]). Denote \mathcal{D}_X as the collection of all compatible metrics on X . Given $d \in \mathcal{D}_X$, for $\omega \in \Omega, \epsilon > 0$ and $n \in \mathbb{N}$, a subset E of $K(\omega)$ is called an $(\omega, n, \epsilon, \phi; d)$ -separated subset of $K(\omega)$ if for any distinct points $x, y \in E$, one has that

$$\max_{0 \leq i \leq n-1} d(\phi(i, \omega)x, \phi(i, \omega)y) > \epsilon.$$

Denote the maximal cardinality of all $(\omega, n, \epsilon, \phi; d)$ -separated subsets of $K(\omega)$ as $r_n(K, \omega, \epsilon, \phi; d)$. The *topological entropy* of (K, ϕ) is defined as following

$$(2.1) \quad h_{top}(\phi, K) := \inf_{d \in \mathcal{D}_X} \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \int_{\Omega} \log r_n(K, \omega, \epsilon, \phi; d) d\mathbb{P}(\omega).$$

Remark 2.5. If X is compact, then

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \int_{\Omega} \log r_n(K, \omega, \epsilon, \phi; d) d\mathbb{P}(\omega)$$

is independent of the choice of metrics on X (for example, see [24, Proposition 2.2]).

We end this subsection by presenting the variational principle for random dynamical systems. The reader can obtain it by combining [19, Lemma A.4] and a similar approach of [24, Proposition 2.2].

Proposition 2.6. *Let (X, ϕ) be a continuous random dynamical system over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and K be a ϕ -invariant random compact set. Then*

$$h_{top}(\phi, K) = \sup\{h_\mu(\phi, K) : \mu \in \mathcal{M}_{\mathbb{P}}^K(\Omega \times X)\} = \sup\{h_\mu(\phi, K) : \mu \in \mathcal{E}_{\mathbb{P}}^K(\Omega \times X)\},$$

where $\mathcal{E}_{\mathbb{P}}^K(\Omega \times X)$ is the set of the ergodic elements of $\mathcal{M}_{\mathbb{P}}^K(\Omega \times X)$.

2.3. Natural extension. In this subsection, we review the natural extension of the MDSs on standard probability spaces. Assume that (X, \mathcal{X}, μ, T) is a MDS on a standard probability space. Let

$$\begin{aligned} \bar{X} &= \{\bar{x} = (x_i)_{i \in \mathbb{Z}} \in X^{\mathbb{Z}} : Tx_i = x_{i+1}, i \in \mathbb{Z}\}, \\ \bar{T} : \bar{X} &\rightarrow \bar{X}, \quad (x_i)_{i \in \mathbb{Z}} \mapsto (Tx_i)_{i \in \mathbb{Z}}, \end{aligned}$$

$\bar{\mathcal{X}}$ be the σ -algebra which is generated by $\bigcup_{n \in \mathbb{Z}} \Pi_{n, X}^{-1} \mathcal{X}$ where $\Pi_{n, X} : \bar{X} \rightarrow X$ with $\Pi_{n, X}(\bar{x}) = x_n$, and $\bar{\mu}$ be the measure on $\bar{\mathcal{X}}$ which is defined by $\bar{\mu}(\Pi_{n, X}^{-1}(A)) = \mu(A)$ for $A \in \mathcal{X}$. It is clear that $(\bar{X}, \bar{\mathcal{X}}, \bar{\mu}, \bar{T})$ is an invertible MDS on a standard probability space. Then

$$\Pi_X := \Pi_{0, X} : (\bar{X}, \bar{\mathcal{X}}, \bar{\mu}, \bar{T}) \rightarrow (X, \mathcal{X}, \mu, T)$$

is a factor map and $(\bar{X}, \bar{\mathcal{X}}, \bar{\mu}, \bar{T})$ is called the natural extension of (X, \mathcal{X}, μ, T) . The reader can refer to [35] for the proof. In [34], it is proved that $(\bar{X}, \bar{\mathcal{X}}, \bar{\mu}, \bar{T})$ is ergodic if and only if (X, \mathcal{X}, μ, T) is ergodic.

Now, we state a result about the conditional measure-theoretic entropy of the natural extension from [19, Lemma 3.2].

Lemma 2.7. Let $\Pi_X : (\bar{X}, \bar{\mathcal{X}}, \bar{\mu}, \bar{T}) \rightarrow (X, \mathcal{X}, \mu, T)$ be the natural extension of the MDS (X, \mathcal{X}, μ, T) on a standard probability space. Then $h_{\bar{\mu}}(\bar{T}|\Pi_X) = 0$.

2.4. Disintegration of measures. In this subsection, we recall some notations and results on the disintegration of measures which are summarized from [13, Chapter 5 and 6].

For a factor map $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ between two invertible MDSs on standard probability spaces, there is a set of conditional probability measures $\{\mu_y\}_{y \in Y}$ with the following properties:

- μ_y is a probability measure on (X, \mathcal{X}) with $\mu_y(\pi^{-1}(y)) = 1$ for ν -a.s. $y \in Y$;
- for each $f \in \mathcal{L}^1(X, \mathcal{X}, \mu)$, one has $f \in \mathcal{L}^1(X, \mathcal{X}, \mu_y)$ for ν -a.s. $y \in Y$, the map $y \mapsto \int_X f d\mu_y$ is in $\mathcal{L}^1(Y, \mathcal{Y}, \nu)$ and $\int_Y (\int_X f d\mu_y) d\nu(y) = \int_X f d\mu$.

$\mu = \int_Y \mu_y d\nu(y)$ is called the *disintegration of μ relative to the factor (Y, \mathcal{Y}, ν, S)* . Furthermore, the measures $\{\mu_y\}_{y \in Y}$ are essentially unique and satisfy that $T_*\mu_y = \mu_{S y}$ for ν -a.s. $y \in Y$. The conditional expectations and the conditional measures are related by

$$(2.2) \quad \mathbb{E}(f|\pi^{-1}\mathcal{Y})(x) = \int_X f d\mu_{\pi(x)} \quad \text{for } \mu\text{-a.s. } x \in X$$

for every $f \in \mathcal{L}^1(X, \mathcal{X}, \mu)$. The product of (X, \mathcal{X}, μ, T) with itself relative to factor (Y, \mathcal{Y}, ν, S) is the MDS

$$(X \times X, \mathcal{X} \times \mathcal{X}, \mu \times_Y \mu, T \times T),$$

where the measure $\mu \times_Y \mu = \int_Y (\mu_y \times \mu_y) d\nu(y)$ is $T \times T$ -invariant and is supported on $R_\pi := \{(x_1, x_2) \in X \times X : \pi(x_1) = \pi(x_2)\}$.

The disintegration of measures has another equivalent form of expression. Specifically, denoting $\mathcal{S} := \pi^{-1}\mathcal{Y}$, $\{\mu_x^{\mathcal{S}}\}_{x \in X}$ is the disintegration of μ relative to \mathcal{S} if the followings hold:

- $\mu_x^{\mathcal{S}}$ is a probability measure on (X, \mathcal{X}) with $\mu_x^{\mathcal{S}}(\pi^{-1}\pi(x)) = 1$ for μ -a.s. $x \in X$;
- for μ -a.s. $x \in X$, one has that for any $x_1, x_2 \in \pi^{-1}\pi(x)$, $\mu_{x_1}^{\mathcal{S}} = \mu_{x_2}^{\mathcal{S}}$;
- for each $f \in \mathcal{L}^1(X, \mathcal{X}, \mu)$, one has that $f \in \mathcal{L}^1(X, \mathcal{S}, \mu_x^{\mathcal{S}})$ for μ -a.s. $x \in X$, the map $x \mapsto \int_X f d\mu_x^{\mathcal{S}}$ belongs to $\mathcal{L}^1(X, \mathcal{X}, \mu)$ and $\int_X (\int_X f d\mu_x^{\mathcal{S}}) d\mu(x) = \int_X f d\mu$.

We remark that the two forms above-mentioned are equivalent. Namely, $\mu_x^{\mathcal{S}} = \mu_{\pi(x)}$ for μ -a.s. $x \in X$.

2.5. Relative Pinsker σ -algebra. In this subsection, we introduce some notations and results about the relative Pinsker σ -algebra.

Let (X, \mathcal{X}, μ, T) be a MDS on a standard probability space and \mathcal{S} be a T -invariant sub- σ -algebra of \mathcal{X} . The *relative Pinsker σ -algebra $\mathcal{P}_\mu(\mathcal{S})$* with respect to \mathcal{S} is defined as the smallest σ -algebra containing

$$\{A \in \mathcal{X} : h_\mu(T, \{A, A^c\}|\mathcal{S}) = 0\}.$$

By [40, Section 4.10], $\mathcal{P}_\mu(\mathcal{S})$ is a T -invariant sub- σ -algebra of \mathcal{X} . Hence, it uniquely (up to an isomorphism) determines a factor (Y, \mathcal{Y}, ν, S) of (X, \mathcal{X}, μ, T) (see [13, Theorem 6.5]). That is, there exists a factor map

$$\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$$

between two MDSs on standard probability spaces such that $\pi^{-1}(\mathcal{Y}) = \mathcal{S} \pmod{\mu}$. Usually, $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ is called *relative Pinsker factor map* with respect to \mathcal{S} . If $\mathcal{S} = \{X, \emptyset\}$, then $\mathcal{P}_\mu(T) := \mathcal{P}_\mu(\mathcal{S})$ is called *Pinsker σ -algebra* of (X, \mathcal{X}, μ, T) .

The following lemma is well-known in MDSs, which will be used in the proof of Theorem 3.8 and the main theorems. The reader can refer to [5, Theorem 2.1 and Theorem 2.3] or [43, Lemma 4.1]).

Lemma 2.8. Assume that $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Z, \mathcal{Z}, \eta, R)$ is a factor map between two ergodic MDSs on standard probability spaces. Let $\pi_1 : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ be the relative Pinsker factor map with respect to $\pi^{-1}\mathcal{Z}$ and $\mu = \int_Y \mu_y d\nu(y)$ be the disintegration of μ relative to the factor (Y, \mathcal{Y}, ν, S) . If $h_\mu(T|\pi) > 0$, then μ_y is non-atomic (i.e. $\mu_y(\{x\}) = 0$ for each $x \in X$) for ν -a.s. $y \in Y$.

Finally, we give a proposition which describes the relation between two different relative Pinsker factors. The reader can see a more general form in [9, Theorem 0.4 (iii)].

Proposition 2.9. Let $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Z, \mathcal{Z}, \eta, R)$ be a factor map between two MDSs on standard probability spaces and $\pi_1 : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ be the relative Pinsker factor map with respect to $\pi^{-1}\mathcal{Z}$. Denoting $\mu = \int_Y \mu_y d\nu(y)$ as the disintegration of μ relative to the factor (Y, \mathcal{Y}, ν, S) and

$$(X \times X, \mathcal{X} \times \mathcal{X}, \mu \times_Y \mu, T \times T),$$

as the product of (X, \mathcal{X}, μ, T) with itself relative to factor (Y, \mathcal{Y}, ν, S) , then one has that

$$(2.3) \quad \mathcal{P}_\lambda((\pi \circ \text{Proj}_1)^{-1}\mathcal{Z}) = (\pi_1 \circ \text{Proj}_1)^{-1}\mathcal{Y} \pmod{\lambda}$$

where $\lambda = \mu \times_Y \mu$ and $\text{Proj}_1 : X \times X \rightarrow X$ is the projection to the first coordinate.

3. TWO FUNDAMENTAL THEOREMS

Letting X be a Borel subset of a Polish space \tilde{X} and $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Z, \mathcal{Z}, \eta, R)$ ¹ be a factor map between two MDSs on standard probability spaces, we prove that the stable sets are dense in the support of conditional measure with respect to the relative Pinsker factor, namely Theorem 3.1. Additionally assuming that $h_\mu(T|\pi) > 0$ and (X, \mathcal{X}, μ, T) is ergodic, one has that the set of all separating pairs has the full measure under the product of conditional measure with respect to the relative Pinsker factor, namely Theorem 3.8.

The reason why we don't directly assume that (X, \mathcal{X}, μ, T) is a Polish system is that when we prove Theorem 1.2, the random compact set may not be a Polish space but only a Borel subset of the whole space.

3.1. Stable sets. The following theorem is a generalization of [42, Lemma 3.2] to our setting. In the proof of it, the main difficulty is that the previous results depend on the compactness of the space. Through some more detail observations, we can prove that, in non-compact systems, there also are many asymptotic pairs.

Theorem 3.1. Let X be a Borel subset of a Polish space \tilde{X} and $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Z, \mathcal{Z}, \eta, R)$ be a factor map between two invertible MDSs on standard probability spaces. Denote $\pi_1 : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ as the Pinsker factor map with respect to $\pi^{-1}\mathcal{Z}$ and $\mu = \int_Y \mu_y d\nu(y)$ as the disintegration of μ relative to (Y, \mathcal{Y}, ν, S) . For any infinite positive integer sequence $\mathbf{a} = \{a_n\}_{n \in \mathbb{N}}$ (i.e. $\lim_{n \rightarrow +\infty} a_n = +\infty$), there exists a μ -full measure subset X_1 such that for any $x \in X_1$,

$$(3.1) \quad \overline{W_{\mathbf{a}}^s(x, T) \cap \text{supp}(\mu_{\pi_1(x)})} = \text{supp}(\mu_{\pi_1(x)})$$

where $W_{\mathbf{a}}^s(x, T) = \{y \in X : \lim_{n \rightarrow +\infty} d(T^{a_n}x, T^{a_n}y) = 0\}$ and d is a compatible complete metric on \tilde{X} .

¹In this place, \mathcal{X} is the σ -algebra generated by the Borel subsets of X .

Remark 3.2. In Theorem 3.1, for a probability measure $\widehat{\mu}$ on (X, \mathcal{X}) , $\text{supp}(\widehat{\mu})$ is the smallest closed subset A of \widetilde{X} with $\widehat{\mu}(A) = 1$. The closure in (3.1) is with respect to the topology of \widetilde{X} .

In order to prove Theorem 3.1, let us begin with some lemmas. The following lemma is a classical result. The reader can refer to [22, Theorem (17.11)] for the proof.

Lemma 3.3. Assume that (X, \mathcal{X}, μ) is a Polish probability space. Then for any $A \in \mathcal{X}$ and $\epsilon > 0$, there exists a compact subset $A_\epsilon \subset A$ such that $\mu(A \setminus A_\epsilon) \leq \epsilon$.

Let (X, \mathcal{X}, μ, T) be a MDS. A partition ξ is called a measurable partition of (X, \mathcal{X}) if

$$\xi = \bigvee_{i \in I} \xi_i$$

where $\{\xi_i\}_{i \in I}$ is a countable family of finite measurable partitions. Denote $\sigma(\xi)$ as the smallest sub- σ -algebra of \mathcal{X} which contains ξ . For two measurable partitions ξ_1 and ξ_2 of (X, \mathcal{X}) , it can be shown that $\sigma(\xi_1) \vee \sigma(\xi_2) = \sigma(\xi_1 \vee \xi_2)$ where $\sigma(\xi_1) \vee \sigma(\xi_2)$ is the smallest sub- σ -algebra of \mathcal{X} that contains the σ -algebras $\sigma(\xi_1)$ and $\sigma(\xi_2)$, so there is no ambiguity to denote $\sigma(\xi_1 \vee \xi_2)$ by $\xi_1 \vee \xi_2$. For convenience, we write $\xi_1 \preceq \xi_2$ if for any element of ξ_2 is contained in some element of ξ_1 . For a measurable partition ξ of (X, \mathcal{X}, μ, T) , put $\xi(x)$ as the element of ξ containing x ,

$$\xi^- = \bigvee_{n \in \mathbb{N}} T^{-n} \xi \quad \text{and} \quad \xi^T = \bigvee_{n \in \mathbb{Z}} T^{-n} \xi.$$

The measurable partition ξ is called *measurable generating partition* if $\xi^T = \mathcal{X} \pmod{\mu}$.

Now, we summary a few results from [43] as the following lemma. Particularly, (i) of Lemma 3.4 is from [43, Lemma 3.1]; (ii) of Lemma 3.4 is from [43, Theorem 3.3, Lemma 3.5]; and (iii) of Lemma 3.4 is from [43, Lemma 3.6].

Lemma 3.4. For an invertible MDS (X, \mathcal{X}, μ, T) on standard probability spaces, let α, β, γ be finite measurable partitions of (X, \mathcal{X}) and \mathcal{S} be a T -invariant sub- σ -algebra of \mathcal{X} . Then, we have

- (i) If $\alpha \preceq \beta$, then $\lim_{n \rightarrow +\infty} H_\mu(\alpha | \beta^- \vee T^{-n} \gamma^- \vee \mathcal{S}) = H_\mu(\alpha | \beta^- \vee \mathcal{S})$.
- (ii) $H_\mu(\alpha | \alpha^- \vee \mathcal{P}_\mu(\mathcal{S})) = H_\mu(\alpha | \alpha^- \vee \mathcal{S}) = h_\mu(T, \alpha | \mathcal{S})$ and

$$h_\mu(T, \alpha \vee \beta | \mathcal{S}) = h_\mu(T, \beta | \mathcal{S}) + h_\mu(T, \alpha | \beta^T \vee \mathcal{S}).$$

- (iii) If ξ is a measurable generating partition of (X, \mathcal{X}) with $\xi \supset \mathcal{S} \pmod{\mu}$, then

$$\bigcap_{n \in \mathbb{N}_0} T^{-n} \xi^- \supset \mathcal{P}_\mu(\mathcal{S}) \pmod{\mu}.$$

Following ideas in [43, Lemma 3.7], we prove the corresponding result for our setting. This is a key lemma to prove Theorem 3.1. We use Lemma 3.3 to construct the expected measurable partition to overcome the difficulty that the phase is not compact.

Lemma 3.5. Let X be a Borel subset of a Polish space \widetilde{X} and (X, \mathcal{X}, μ, T) be an invertible MDS and \mathcal{S} be a T -invariant sub- σ -algebra of \mathcal{X} . Then (X, \mathcal{X}, T, μ) admits a measurable generating partition α of (X, \mathcal{X}) with following properties:

- (a) $\alpha \supset \mathcal{S} \pmod{\mu}$ and $\mathcal{P}_\mu(\mathcal{S}) = \bigcap_{n \in \mathbb{N}_0} T^{-n} \alpha^- \pmod{\mu}$,
- (b) the set

$$\{x \in X : \text{any pair of points belonging to } \alpha^-(x) \text{ is asymptotic along } \mathbb{N}^1\}$$

has μ -full measure.

¹A pair $(x, y) \in X \times X$ is called asymptotic along $\mathbf{a} = \{a_n\}_{n \in \mathbb{N}}$ if $\lim_{n \rightarrow +\infty} d(T^{a_n} x, T^{a_n} y) = 0$, where d is the compatible complete metric on \widetilde{X} ,

Proof. By Lemma 3.3, we can find a sequence of compact subsets $\{X_n\}_{n \in \mathbb{N}}$ of X with $\mu(X_n) \geq 1 - \frac{1}{2^n}$ and $X_n \subset X_{n+1}$ for any $n \in \mathbb{N}$. Therefore, we can choose a finite measurable partition ξ_n of (X_n, \mathcal{X}_n) for each $n \in \mathbb{N}$ where $\mathcal{X}_n = \{A \cap X_n : A \in \mathcal{X}\}$, with the following properties:

- (I) $\xi_n \preceq \xi_{n+1}$ for each $n \in \mathbb{N}$;
- (II) the maximal diameter of ξ_n goes to zero as $n \rightarrow +\infty$, namely, $\lim_{n \rightarrow +\infty} \max\{\text{diam}(A) : A \in \xi_n\} = 0$.

For each $n \in \mathbb{N}$, denoting $\mathcal{U}_n = \xi_n \cup \{X \setminus X_n\}$ which is a finite measurable partition of (X, \mathcal{X}) , then $\mathcal{U}_{n+1} \succeq \mathcal{U}_n$ and $\bigvee_{n \in \mathbb{N}} \mathcal{U}_n = \mathcal{X} \pmod{\mu}$.

Define inductively $\mathcal{V}_n = \mathcal{V}_{n-1} \vee T^{-t_{n-1}} \mathcal{U}_{n-1}$ for $n \geq 2$, where $\mathcal{V}_1 = \mathcal{U}_1$ and $t_n \in \mathbb{N}$ are to be chosen by using (i) of Lemma 3.4 such that

$$H_\mu(\mathcal{V}_m | \mathcal{V}_{n-1}^- \vee \mathcal{S}) - H_\mu(\mathcal{V}_m | \mathcal{V}_n^- \vee \mathcal{S}) < \frac{1}{m} \frac{1}{2^{n-m}}, \quad m = 1, 2, \dots, n-1.$$

Fixing $m \in \mathbb{N}$, for any $t > m$, one has that

$$(3.2) \quad \begin{aligned} & H_\mu(\mathcal{V}_m | \mathcal{V}_m^- \vee \mathcal{S}) - H_\mu(\mathcal{V}_m | \mathcal{V}_t^- \vee \mathcal{S}) \\ &= \sum_{i=m}^{t-1} (H_\mu(\mathcal{V}_m | \mathcal{V}_i^- \vee \mathcal{S}) - H_\mu(\mathcal{V}_m | \mathcal{V}_{i+1}^- \vee \mathcal{S})) \leq \frac{1}{m}. \end{aligned}$$

Clearly, $\beta = \bigvee_{n \in \mathbb{N}} \mathcal{V}_n$ is a measurable generating partition of (X, \mathcal{X}) . Due to (3.2), one has that

$$(3.3) \quad \lim_{m \rightarrow +\infty} (H_\mu(\mathcal{V}_m | \mathcal{V}_m^- \vee \mathcal{S}) - H_\mu(\mathcal{V}_m | \beta^- \vee \mathcal{S})) = 0.$$

In the next, we will explain that

$$(3.4) \quad \bigcap_{n \in \mathbb{N}_0} (T^{-n} \beta^- \vee \mathcal{S}) \subset \mathcal{P}_\mu(\mathcal{S}) \pmod{\mu}.$$

In fact, we only need to illustrate that for any a finite measurable partition ξ of (X, \mathcal{X}) with $\sigma(\xi) \subset \bigcap_{n \in \mathbb{N}_0} (T^{-n} \beta^- \vee \mathcal{S})$, one has that $\sigma(\xi) \subset \mathcal{P}_\mu(\mathcal{S}) \pmod{\mu}$. By (ii) of Lemma 3.4 and $\xi^T \subset \beta^- \vee \mathcal{S}$, one has

$$\begin{aligned} H_\mu(\xi | \xi^- \vee \mathcal{S}) &= H_\mu(\mathcal{V}_m \vee \xi | \mathcal{V}_m^- \vee \xi^- \vee \mathcal{S}) - H_\mu(\mathcal{V}_m | \mathcal{V}_m^- \vee \xi^T \vee \mathcal{S}) \\ &\leq H_\mu(\xi | \mathcal{V}_m^- \vee \mathcal{S}) + H_\mu(\mathcal{V}_m | \mathcal{V}_m^- \vee \mathcal{S}) - H_\mu(\mathcal{V}_m | \beta^- \vee \mathcal{S}). \end{aligned}$$

As $m \rightarrow +\infty$, by (3.3) we have

$$h_\mu(T, \xi | \mathcal{S}) = H_\mu(\xi | \xi^- \vee \mathcal{S}) \leq H_\mu(\xi | \beta^- \vee \mathcal{S}) = 0,$$

which implies the (3.4).

Take a measurable partition γ of (X, \mathcal{X}) such that $\sigma(\gamma) = \mathcal{S} \pmod{\mu}$. Then $\alpha = \beta \vee \gamma$ is the required measurable partition. Indeed, α is a measurable generating partition with $\sigma(\alpha) \supseteq \mathcal{S} \pmod{\mu}$. By (iii) of Lemma 3.4, we have $\mathcal{P}_\mu(\mathcal{S}) \subset \bigcap_{n \in \mathbb{N}_0} T^{-n} \alpha^- \pmod{\mu}$. On the other hand, by Claim 3.4, one has that

$$\bigcap_{n \in \mathbb{N}_0} T^{-n} \alpha^- = \bigcap_{n \in \mathbb{N}_0} (T^{-n} \beta^- \vee T^{-n} \gamma^-) \subset \bigcap_{n \in \mathbb{N}_0} (T^{-n} \beta^- \vee \mathcal{S}) \subset \mathcal{P}_\mu(\mathcal{S}) \pmod{\mu}.$$

Therefore, $\mathcal{P}_\mu(\mathcal{S}) = \bigcap_{n \in \mathbb{N}_0} T^{-n} \alpha^- \pmod{\mu}$. Note that if $x \notin \bigcup_{m \in \mathbb{N}} \bigcap_{j \in \mathbb{N}} \bigcup_{i \geq j} T^{-m-t_k}(X \setminus X_i)$, then for any pair of points belonging to $\alpha^-(x)$ is asymptotic along \mathbb{N} . Indeed, since

$$\bigvee_{m \in \mathbb{N}} T^{-m} \alpha \succeq \bigvee_{m \in \mathbb{N}} \bigvee_{j \in \mathbb{N}} T^{-m} \mathcal{V}_j \succeq \bigvee_{m \in \mathbb{N}} \bigvee_{j \in \mathbb{N}} T^{-m-t_j} \mathcal{U}_j,$$

there exist $A_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $x \in \alpha^-(x) \subset \bigcap_{m \in \mathbb{N}} \bigcap_{j \in \mathbb{N}} T^{-m-t_j} A_j$. By assumption that $x \notin \bigcup_{m \in \mathbb{N}} \bigcap_{j \in \mathbb{N}} \bigcup_{i \geq j} T^{-m-t_i} (X \setminus X_i)$, there exists a strictly monotone increasing positive integer sequence $\{k_n\}_{n=1}^\infty$ such that $A_{k_n} \in \xi_{k_n}$ for each $n \in \mathbb{N}$. Due to (II), we know that any pair of points belonging to $\alpha^-(x)$ is asymptotic along \mathbb{N} .

Finally, combining the following fact that

$$\begin{aligned} \mu \left(\bigcup_{m \in \mathbb{N}} \bigcap_{j \in \mathbb{N}} \bigcup_{i \geq j} T^{-m-t_i} (X \setminus X_i) \right) &\leq \sum_{m \in \mathbb{N}} \mu \left(\bigcap_{j \in \mathbb{N}} \bigcup_{i \geq j} T^{-m-t_i} X_i^c \right) \\ &= \sum_{m \in \mathbb{N}} \lim_{j \rightarrow +\infty} \mu \left(\bigcup_{i \geq j} T^{-m-t_i} X_i^c \right) \\ &\leq \sum_{m \in \mathbb{N}} \lim_{j \rightarrow +\infty} \sum_{i \geq j} \frac{1}{2^i} = 0, \end{aligned}$$

we finish the proof of Lemma 3.5. \square

The following result is a corollary of [13, Corollary 5.21] and [8, Theorem 3.17]. For completeness, we present a direct proof here.

Lemma 3.6. Let (X, \mathcal{X}, μ) be a Polish probability space, $\{\mathcal{S}_n\}_{n \in \mathbb{N}}$ be a decreasing sub- σ -algebra of \mathcal{X} and $\mathcal{S} = \bigcap_{n \in \mathbb{N}} \mathcal{S}_n$. Denote $\mu = \int_X \mu_{n,x} d\mu(x)$ and $\mu_\infty = \int_X \mu_{\infty,x} d\mu(x)$ as the disintegrations relative to \mathcal{S}_n and \mathcal{S} , respectively. Then for μ -a.s. $x \in X$, the following inequality

$$(3.5) \quad \liminf_{n \rightarrow +\infty} \mu_{n,x}(U) \geq \mu_{\infty,x}(U)$$

holds for any open subset U of X .

Proof. Let $\{A_n\}_{n \in \mathbb{N}}$ be a countable topological basis of X . By Theorem 2.2, there exists μ -full measure subset X_1 such that for any $x \in X_1$, we have

$$(3.6) \quad \lim_{n \rightarrow +\infty} \mu_{n,x} \left(\bigcup_{i \in I} A_i \right) = \mu_{\infty,x} \left(\bigcup_{i \in I} A_i \right)$$

for any finite subset $I \subset \mathbb{N}$. For any $x \in X_1$ and open subset $U := \bigcup_{i \in \mathbb{N}} A_{k_i}$, one has

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \mu_{n,x}(U) &= \liminf_{n \rightarrow +\infty} \mu_{n,x} \left(\bigcup_{i \in \mathbb{N}} A_{k_i} \right) \\ &\geq \liminf_{m \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \mu_{n,x} \left(\bigcup_{i=1}^m A_{k_i} \right) \\ &= \liminf_{m \rightarrow +\infty} \mu_{\infty,x} \left(\bigcup_{i=1}^m A_{k_i} \right) \\ &= \mu_{\infty,x}(U). \end{aligned}$$

This finishes the proof of Lemma 3.6. \square

The following lemma is from [17, Lemma 2.1]. The only difference between Lemma 3.7 and [17, Lemma 2.1] is whether the phase space X is compact. In fact, the proof of [17, Lemma 2.1] still holds for non-compact space. So, we omit the proof here.

Lemma 3.7. Let (X, \mathcal{X}, μ) a Polish probability space and $\mathcal{S}_2 \subset \mathcal{S}_1$ be two sub- σ -algebras of \mathcal{X} . Denote $\mu = \int_X \mu_{i,x} d\mu(x)$ as the disintegration of μ relative to \mathcal{S}_i for $i = 1, 2$, respectively. Then $\text{supp}(\mu_{1,x}) \subset \text{supp}(\mu_{2,x})$ for μ -a.s. $x \in X$.

With the help of lemmas above-mentioned, we give the proof of Theorem 3.1 as follows.

Proof of Theorem 3.1. Recall that X is a Borel subset of a Polish space \tilde{X} , $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Z, \mathcal{Z}, \eta, R)$ is a factor map between two MDSs on standard probability spaces, $\pi_1 : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ is the Pinsker factor map with respect to $\pi^{-1}\mathcal{Z}$, and $\mu = \int_Y \mu_y d\nu(y)$ is the disintegration of μ relative to the factor (Y, \mathcal{Y}, ν, S) .

Fix an infinite positive integer sequence $\mathbf{a} = \{a_n\}_{n \in \mathbb{N}}$. By Lemma 3.5, there exists a measurable partition α such that

$$\mathcal{P}_\mu(\pi^{-1}\mathcal{Z}) = \bigcap_{n \in \mathbb{N}_0} T^{-n}\alpha^- \pmod{\mu}$$

and for μ -a.s. $x \in X$, any pair of points belonging to $\alpha^-(x)$ is asymptotic along \mathbb{N} . Hence $(T^{-n}\alpha^-)(x) \subset W_{\mathbf{a}}^s(x, T)$ for each $n \in \mathbb{N}_0$ and μ -a.s. $x \in X$. For any $n \in \mathbb{N}_0$, let $\mu = \int_X \mu_{n,x} d\mu(x)$ be the disintegration of μ relative to $T^{-n}\alpha^-$. Following from the definition of the measure disintegration, for any $n \in \mathbb{N}_0$ and μ -a.s. $x \in X$, one has that

$$(3.7) \quad \mu_{n,x}(W_{\mathbf{a}}^s(x, T)) = 1.$$

Note that

$$\alpha^- \supset T^{-1}\alpha^- \supset T^{-2}\alpha^- \supset \cdots \text{ and } \bigcap_{n \in \mathbb{N}_0} T^{-n}\alpha^- = \mathcal{P}_\mu(\pi^{-1}\mathcal{Z}) \pmod{\mu}.$$

Applying Lemma 3.6 and Lemma 3.7 on $\{T^{-n}\alpha^-\}_{n \in \mathbb{N}_0}$ and $\mathcal{P}_\mu(\pi^{-1}\mathcal{Z})$ for the Polish probability space $(\tilde{X}, \tilde{\mathcal{X}}, \mu)$ where $\tilde{\mathcal{X}}$ is the Borel- σ algebra of \tilde{X} , there exists a μ -full measure subset $X' \subset X$ such that for any $x \in X'$, any closed subset F of \tilde{X} and any $m \in \mathbb{N}$, one has that

$$(3.8) \quad \limsup_{n \rightarrow +\infty} \mu_{n,x}(F) \leq \mu_{\pi_1(x)}(F),$$

and

$$(3.9) \quad \text{supp}(\mu_{m,x}) \subseteq \text{supp}(\mu_{m+1,x}) \subseteq \cdots \subseteq \text{supp}(\mu_{\pi_1(x)}).$$

By (3.7) and (3.9), one has that for $x \in X'$ and any $n \in \mathbb{N}_0$,

$$\mu_{n,x}(\overline{W_{\mathbf{a}}^s(x, T) \cap \text{supp}(\mu_{\pi_1(x)})}) \geq \mu_{n,x}(W_{\mathbf{a}}^s(x, T) \cap \text{supp}(\mu_{\pi_1(x)})) = 1,$$

which implies that

$$\mu_{\pi_1(x)}(\overline{W_{\mathbf{a}}^s(x, T) \cap \text{supp}(\mu_{\pi_1(x)})}) \stackrel{(3.8)}{\geq} \limsup_{n \rightarrow +\infty} \mu_{n,x}(\overline{W_{\mathbf{a}}^s(x, T) \cap \text{supp}(\mu_{\pi_1(x)})}) = 1.$$

Therefore, $\text{supp}(\mu_{\pi_1(x)}) \subseteq \overline{W_{\mathbf{a}}^s(x, T) \cap \text{supp}(\mu_{\pi_1(x)})}$.

On the other hand, as for any $x \in X'$, it is clear that $\text{supp}(\mu_{\pi_1(x)}) \supseteq \overline{W_{\mathbf{a}}^s(x, T) \cap \text{supp}(\mu_{\pi_1(x)})}$. This completes the proof of Theorem 3.1. \square

3.2. Separating pairs. In this subsection, borrowing the ideas in [31], we prove Theorem 3.8. In the proof of [31, Theorem 1.4], the similar result has been established for non-relative factor case in compact systems.

Theorem 3.8. *Let X be a Borel subset of a Polish space \tilde{X} and $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Z, \mathcal{Z}, \eta, R)$ be a factor map between two invertible ergodic MDSs on standard probability spaces. Denote $\pi_1 : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ as the relative Pinsker factor map with respect to $\pi^{-1}\mathcal{Z}$, $\mu = \int_Y \mu_y d\nu(y)$ as the disintegration of μ relative to the factor (Y, \mathcal{Y}, ν, S) . If $h_\mu(T|\pi) > 0$, then for any infinite positive integer sequence $\mathbf{a} = \{a_i\}_{i \in \mathbb{N}}$ (i.e. $\lim_{i \rightarrow +\infty} a_i = +\infty$), there exists a ν -full measure set Y_0 such that for any $y \in Y_0$, there exists a positive constant δ_y such that*

$$\mu_y \times \mu_y(\mathcal{W}_{\mathbf{a}}(X, \delta_y)) = 1,$$

where

$$\mathcal{W}_{\mathbf{a}}(X, \delta_y) = \left\{ (x_1, x_2) \in X \times X : \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n d(T^{a_i} x_1, T^{a_i} x_2) > \delta_y \right\},$$

d is a compatible complete metric on \tilde{X} .

To prove Theorem 3.8, we recall the notion of the characteristic σ -algebra firstly, and then prove that relative Pinsker algebra is the characteristic σ -algebra along any infinite positive integer sequence for MDSs, namely Proposition 3.11.

Definition 3.9. Let (X, \mathcal{X}, μ, T) be a MDS. A sub- σ -algebra \mathcal{S} of \mathcal{X} is called a characteristic σ -algebra for the positive integer sequence $\mathbf{a} = \{a_n\}_{n \in \mathbb{N}}$ if for any $f \in \mathcal{L}^2(X, \mathcal{X}, \mu)$,

$$(3.10) \quad \lim_{N \rightarrow +\infty} \|\mathbf{A}(\mathbf{a}_N, f) - \mathbf{A}(\mathbf{a}_N, \mathbb{E}(f|\mathcal{S}))\|_2 = 0,$$

where $\mathbf{A}(\mathbf{a}_N, f) = \frac{1}{N} \sum_{i=1}^N f \circ T^{a_i}$ and $\|f\|_2 := \left(\int_X |f|^2 d\mu \right)^{1/2}$.

In [11, 31], it was proved that Pinsker algebra is the characteristic σ -algebra along any infinite sequence in countable discrete amenable groups in compact metric spaces. We restate a convenient version of this theorem as follows.

Lemma 3.10. Let (X, \mathcal{X}, μ, T) be an invertible MDS on a standard probability space. If $\mathbf{a} = \{a_n\}_{n \in \mathbb{N}}$ is an infinite positive integer sequence, then the Pinsker algebra of (X, \mathcal{X}, μ, T) is a characteristic σ -algebra for \mathbf{a} .

The corresponding result of Lemma 3.10 also holds for relative Pinsker σ -algebra. Specifically,

Proposition 3.11. *Let $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Z, \mathcal{Z}, \eta, R)$ be an invertible factor map between two MDSs on standard probability spaces. If $\mathbf{a} = \{a_n\}_{n \in \mathbb{N}}$ is an infinite positive integer sequence, then $\mathcal{P}_\mu(\pi^{-1}\mathcal{Z})$ is a characteristic σ -algebra for \mathbf{a} .*

Proof. Let $\mathcal{P}_\mu(T)$ be the Pinsker σ -algebra of (X, \mathcal{X}, μ, T) . Then

$$\mathcal{P}_\mu(T) \subset \mathcal{P}_\mu(\pi^{-1}\mathcal{Z}),$$

which implies that for any $f, g \in \mathcal{L}^2(X, \mathcal{X}, \mu)$,

$$\langle f - \mathbb{E}(f|\mathcal{P}_\mu(\pi^{-1}\mathcal{Z})), \mathbb{E}(g|\mathcal{P}_\mu(\pi^{-1}\mathcal{Z})) - \mathbb{E}(g|\mathcal{P}_\mu(T)) \rangle = 0.$$

Hence, for any $N \in \mathbb{N}$,

$$\begin{aligned} & \left(\|\mathbf{A}(\mathbf{a}_N, f) - \mathbf{A}(\mathbf{a}_N, \mathbb{E}(f|\mathcal{P}_\mu(T)))\|_2 \right)^2 \\ &= \left(\|\mathbf{A}(\mathbf{a}_N, f) - \mathbf{A}(\mathbf{a}_N, \mathbb{E}(f|\mathcal{P}_\mu(\pi^{-1}\mathcal{Z}))\|_2 \right)^2 \\ & \quad + \left(\|\mathbf{A}(\mathbf{a}_N, \mathbb{E}(f|\mathcal{P}_\mu(\pi^{-1}\mathcal{Z})) - \mathbf{A}(\mathbf{a}_N, \mathbb{E}(f|\mathcal{P}_\mu(T)))\|_2 \right)^2. \end{aligned}$$

That is,

$$\|\mathbf{A}(\mathbf{a}_N, f) - \mathbf{A}(\mathbf{a}_N, \mathbb{E}(f|\mathcal{P}_\mu(\pi^{-1}\mathcal{Z}))\|_2 \leq \|\mathbf{A}(\mathbf{a}_N, f) - \mathbf{A}(\mathbf{a}_N, \mathbb{E}(f|\mathcal{P}_\mu(T)))\|_2.$$

By Lemma 3.10, one has that

$$\lim_{N \rightarrow +\infty} \|\mathbf{A}(\mathbf{a}_N, f) - \mathbf{A}(\mathbf{a}_N, \mathbb{E}(f|\mathcal{P}_\mu(\pi^{-1}\mathcal{Z}))\|_2 = 0.$$

We finish the proof of Proposition 3.11. \square

To prove Theorem 3.8, we recall the notion of the weak convergence and some results about it. Let (X, \mathcal{X}, μ) be a probability space. A sequence $\{h_n\}_{n \in \mathbb{N}}$ in $\mathcal{L}^2(X, \mathcal{X}, \mu)$ converges weakly to $h \in \mathcal{L}^2(X, \mathcal{X}, \mu)$ (denoted by $h_n \xrightarrow{w} h$), if $\lim_{n \rightarrow +\infty} \int h_n f d\mu = \int h f d\mu$ for all $f \in \mathcal{L}^2(X, \mathcal{X}, \mu)$. The following results are proved in [31, Lemma 4.1 and Lemma 4.2].

Lemma 3.12. Let (X, \mathcal{X}, μ, T) be a MDS and $\mathbf{a} = \{a_n\}_{n \in \mathbb{N}}$ be an infinite positive integer sequence. Given a T -invariant sub- σ -algebra \mathcal{S} of \mathcal{X} and $f \in \mathcal{L}^2(X, \mathcal{X}, \mu)$, assume that there exists $P(f) \in \mathcal{L}^2(X, \mathcal{X}, \mu)$ such that $\mathbf{A}(\mathbf{a}_n, f) \xrightarrow{w} P(f)$. Then,

(i) if $\lim_{n \rightarrow +\infty} \|\mathbf{A}(\mathbf{a}_n, f) - \mathbf{A}(\mathbf{a}_n, \mathbb{E}(f|\mathcal{S}))\|_2 = 0$, one has that

$$\mathbf{A}(\mathbf{a}_n, \mathbb{E}(f|\mathcal{S})) \xrightarrow{w} P(f),$$

as $n \rightarrow \infty$. Moreover, $P(f) \in \mathcal{L}^2(X, \mathcal{S}, \mu)$.

(ii) if f is real-valued, then $\limsup_{n \rightarrow +\infty} \mathbf{A}(\mathbf{a}_n, f)(x) \geq P(f)(x)$ for μ -a.s. $x \in X$.

Lemma 3.13. Let (X, \mathcal{X}, μ, T) be a MDS and $\mathbf{a} = \{a_n\}_{n \in \mathbb{N}}$ be an infinite positive integer sequence. Given $g \in \mathcal{L}^2(X, \mathcal{X}, \mu)$ with $g(x) > 0$ for μ -a.s. $x \in X$ and $A \in \mathcal{X}$ with $\mu(A) > 0$, assume that $f = g1_A$, where 1_A is the character function of A , and $\mathbf{A}(\mathbf{a}_n, f) \xrightarrow{w} P(f) \in \mathcal{L}^2(X, \mathcal{X}, \mu)$. Then $P(f)(x) > 0$ for μ -a.s. $x \in A$.

Proof. Assuming that above lemma doesn't hold, then $\mu(B) > 0$, where $B := \{x \in A : P(f)(x) \leq 0\}$. Since $f(x) > 0$ for μ -a.s. $x \in A$, there exists $\epsilon > 0$ such that $\mu(C_\epsilon) < \mu(B)$, where $C_\epsilon = \{x \in A : f(x) \leq \epsilon\}$. For any $n \in \mathbb{N}$,

$$\int_X (f \circ T^n) 1_B d\mu = \int_X f 1_{T^{-n}B} d\mu \geq \int_{T^{-n}B \setminus C_\epsilon} f d\mu \geq \epsilon(\mu(B) - \mu(C_\epsilon)),$$

which implies that

$$\int_X P(f) 1_B d\mu = \lim_{n \rightarrow +\infty} \int_X \mathbf{A}(\mathbf{a}_n, f) 1_B d\mu \geq \epsilon(\mu(B) - \mu(C_\epsilon)) > 0.$$

This contradicts the fact that $\int_X P(f) 1_B d\mu \leq 0$. \square

Lemma 3.14. Assume that $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ is a factor map between two MDSs on standard probability spaces and $A \in \mathcal{X}$ with $\mu(A) > 0$. Denoting $\mu = \int_Y \mu_y d\nu(y)$ as the disintegration of μ relative to the factor (Y, \mathcal{Y}, ν, S) , then there is a subset $B \in \mathcal{Y}$ with $\nu(B) \geq \mu(A)$ such that any $y \in B$, $\mu_y(\pi^{-1}(y) \cap A) > 0$.

Proof. Let $B = \{y \in Y : \mu_y(A) > 0\}$. Then $B \in \mathcal{Y}$ and

$$\nu(B) = \int_B 1 d\nu(y) \geq \int_B \mu_y(A) d\nu(y) = \int_Y \mu_y(A) d\nu(y) = \mu(A).$$

By the definition of measure disintegration, it is clear that for any $y \in B$, $\mu_y(\pi^{-1}(y) \cap A) > 0$. \square

After all preparations, we begin to prove Theorem 3.8.

Proof of Theorem 3.8. Recall that X is a Borel subset of a Polish space \tilde{X} , $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Z, \mathcal{Z}, \eta, R)$ is a factor map between two MDSs on standard probability spaces, $\pi_1 : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ is the relative Pinsker factor map with respect to $\pi^{-1}\mathcal{Z}$, and d is a compatible complete metric on \tilde{X} . Let $\mu = \int_{\mathcal{Y}} \mu_y d\nu(y)$ be the disintegration of μ relative to (Y, \mathcal{Y}, ν, S) .

Claim 3.15. Given a compact subset X_1 of X with $\mu(X_1) > 0$ and an infinite positive integer sequence $\mathbf{a} = \{a_i\}_{i \in \mathbb{N}}$, there exists a measurable set Y_1 with $\nu(Y_1) \geq \mu \times_Y \mu(X_1 \times X_1)$ such that for any $y \in Y_1$, there exists $\delta_y(1) > 0$ such that

$$\mu_y \times \mu_y(\mathcal{W}_{\mathbf{a}}(X_1, \delta_y(1))) = \mu_y \times \mu_y(X_1 \times X_1) > 0,$$

where

$$\mathcal{W}_{\mathbf{a}}(X_1, \delta_y(1)) = \left\{ (x_1, x_2) \in X_1 \times X_1 : \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n f_1(T^{a_i} x_1, T^{a_i} x_2) > \delta_y(1) \right\},$$

and $f_1(x_1, x_2) = 1_{X_1 \times X_1}(x_1, x_2)d(x_1, x_2)$.

Proof. Let $\lambda := \mu \times_Y \mu$ (see Section 2.4) and $\Delta_X = \{(x, x) : x \in X\}$. By Lemma 2.8 and Fubini's theorem, we have

$$\lambda(\Delta_X) = \int_Y \int_X \mu_y(x) d\mu_y(x) d\nu(y) = 0,$$

which implies that $d(x_1, x_2) > 0$ for λ -a.s. $(x_1, x_2) \in X \times X$. And hence that $f_1(x_1, x_2) > 0$ for λ -a.s. $(x_1, x_2) \in X_1 \times X_1$.

Denoting $\mathbf{A}(\mathbf{a}_N, f_1) := \frac{1}{N} \sum_{i=1}^N f_1 \circ (T \times T)^{a_i}$ for any $N \in \mathbb{N}$, it is clear that $\{\mathbf{A}(\mathbf{a}_N, f_1)\}_{N \in \mathbb{N}}$ is a sequence of bounded functions in $\mathcal{L}^2(X \times X, \mathcal{X} \times \mathcal{X}, \lambda)$. By using Alaoglu's theorem, there exists a monotone increasing positive integer sequence $\{a_{k_N}\}_{N \in \mathbb{N}}$ and $P(f_1) \in \mathcal{L}^2(X \times X, \mathcal{X} \times \mathcal{X}, \lambda)$ such that

$$(3.11) \quad \mathbf{A}(\mathbf{a}_{k_N}, f_1) \xrightarrow{w} P(f_1) \text{ as } N \rightarrow \infty.$$

Applying Proposition 3.11 to the $\pi \circ \text{Proj}_1 : (X \times X, \mathcal{X} \times \mathcal{X}, \lambda, T \times T) \rightarrow (Z, \mathcal{Z}, \eta, R)$, where $\text{Proj}_1 : X \times X \rightarrow X$ is the projection to the first coordinate, we have

$$\lim_{N \rightarrow +\infty} \|\mathbf{A}(\mathbf{a}_N, f_1) - \mathbf{A}(\mathbf{a}_N, \mathbb{E}(f_1 | \mathcal{P}_\lambda((\pi \circ \text{Proj}_1)^{-1}\mathcal{Z}))\|_2 = 0.$$

Due to (3.11) and Lemma 3.12, one has

$$\mathbf{A}(\mathbf{a}_{k_N}, \mathbb{E}(f_1 | \mathcal{P}_\lambda((\pi_1 \circ \text{Proj}_1)^{-1}\mathcal{Z}))) \xrightarrow{w} P(f_1) \in \mathcal{L}^2(X \times X, \mathcal{P}_\lambda((\pi_1 \circ \text{Proj}_1)^{-1}\mathcal{Z}), \lambda),$$

as $N \rightarrow \infty$. By using Proposition 2.9, one has that

$$\begin{aligned} P(f_1)(x_1, x_2) &= \mathbb{E}(P(f_1) | \mathcal{P}_\lambda((\pi_1 \circ \text{Proj}_1)^{-1}\mathcal{Z}))(x_1, x_2) \\ &= \mathbb{E}(P(f_1) | (\pi_1 \circ \text{Proj}_1)^{-1}\mathcal{Y})(x_1, x_2) = \int_{X \times X} P(f_1) d(\mu_y \times \mu_y) \end{aligned}$$

for λ -a.s. $(x_1, x_2) \in X \times X$, where $y = \pi_1 \circ \text{Proj}_1((x_1, x_2))$. Combining Lemma 3.12 and Lemma 3.13, there exists a measurable subset B of $X_1 \times X_1$ with $\lambda(B) = \lambda(X_1 \times X_1)$ such that for any $(x_1, x_2) \in B$, we have

$$\limsup_{N \rightarrow +\infty} \mathbf{A}(\mathbf{a}_N, f_1)(x_1, x_2) \geq P(f_1)(x_1, x_2) = \int_{X \times X} P(f_1) d(\mu_y \times \mu_y) := 2\delta_y(1) > 0,$$

where $y = \pi_1 \circ \text{Proj}_1((x_1, x_2))$.

According to Lemma 3.14, there exists $Y'_1 \in \mathcal{Y}$ with $\nu(Y'_1) \geq \lambda(B) = \lambda(X_1 \times X_1)$ such that for any $y \in Y'_1$,

$$\mu_y \times \mu_y((\pi_1 \circ \text{Proj}_1)^{-1}(y) \cap B) > 0.$$

As $\lambda((X_1 \times X_1) \setminus B) = 0$, there exists ν -full measure Y_2'' , such that for any $y \in Y_2''$

$$\mu_y \times \mu_y((X_1 \times X_1) \setminus B) = 0.$$

Let $Y_1 = Y_1' \cap Y_2''$. Then $\nu(Y_1) \geq \lambda(X_1 \times X_1)$. Since $(\pi_1 \circ \text{Proj}_1)^{-1}(y) \cap B \subset \mathcal{W}_{\mathbf{a}}(X_1, \delta_y(1))$ for any $y \in Y_1$, it follows that for each $y \in Y_1$,

$$\begin{aligned} \mu_y \times \mu_y(\mathcal{W}_{\mathbf{a}}(X_1, \delta_y(1))) &\geq \mu_y \times \mu_y((\pi_1 \circ \text{Proj}_1)^{-1}(y) \cap B) \\ &= \mu_y \times \mu_y(B) = \mu_y \times \mu_y(X_1 \times X_1). \end{aligned}$$

This finishes the proof of Claim 3.15. \square

According to Lemma 3.3, there exists a sequence of compact subsets $\{X_n\}_{n \in \mathbb{N}}$ of X with $\mu(X_n) \geq 1 - 1/n$ and $X_n \subset X_{n+1}$ for each $n \in \mathbb{N}$. Applying Claim 3.15 on each X_n , we can choose a subsequence a monotone increasing positive integer sequence $\{a_{k_N}\}_{N \in \mathbb{N}}$ such that for any $n \in \mathbb{N}$,

$$\mathbf{A}(\mathbf{a}_{k_N}, f_n) \xrightarrow{w} P(f_n) \text{ as } N \rightarrow \infty.$$

Furthermore, one has that $\delta_y(n) \geq \delta_y(m)$ if $y \in Y_n \cap Y_m$ and $n \geq m$. Indeed, we only need to prove that $P(f_n) \geq P(f_m)$ for any $n \geq m$. For any $g \in \mathcal{L}^2(X, \mathcal{X}, \mu)$ and $g \geq 0$,

$$\langle P(f_n) - P(f_m), g \rangle = \lim_{N \rightarrow +\infty} \langle \mathbf{A}(\mathbf{a}_{k_N}, f_n) - \mathbf{A}(\mathbf{a}_{k_N}, f_m), g \rangle \geq 0,$$

since $f_n \geq f_m$. Let $Y' = \bigcap_{i \in \mathbb{N}} \bigcup_{n \geq i} Y_n$. Then, by Claim 3.15 and the fact above, one has $\nu(Y') = 1$. For each $y \in Y'$, assume that $y \in \bigcap_{n \in \mathbb{N}} Y_{l_n}$. Then, one has

$$\begin{aligned} \mu_y \times \mu_y(\mathcal{W}_{\mathbf{a}}(X, \delta_y)) &= \liminf_{n \rightarrow +\infty} \mu_y \times \mu_y(\mathcal{W}_{\mathbf{a}}(X_{l_n}, \delta_y)) \\ &\geq \liminf_{n \rightarrow +\infty} \mu_y \times \mu_y(\mathcal{W}_{\mathbf{a}}(X_{l_n}, \delta_y(l_n))) \\ &= \liminf_{n \rightarrow +\infty} \mu_y \times \mu_y(X_{l_n} \times X_{l_n}) = 1, \end{aligned}$$

where $\delta_y := \min\{\delta_y(n) : y \in Y_n\}$. Therefore, Theorem 3.8 holds. \square

4. PROOF OF THE MAIN THEOREMS

In this section, we prove Theorem 1.2 and Theorem 1.1 by using Theorem 3.1 and Theorem 3.8. Recall a result (see [33, Theorem 1]) which is crucial to find Cantor subsets in a perfect complete metric space, firstly.

Lemma 4.1. Let Y be a perfect complete metric space and C be a dense G_δ subset of $Y \times Y$. Then there exists a dense Mycielski subset $S \subseteq Y$ such that $S \times S \subseteq C \cup \Delta_Y$, where $\Delta_Y = \{(y, y) : y \in Y\}$.

4.1. Proof of Theorem 1.2. In this subsection, we assume that (X, ϕ) is an injective continuous random dynamical system over an invertible ergodic Polish system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. In this case, for any ϕ -invariant random compact set K and for any $\mu \in \mathcal{M}_{\mathbb{P}}^K(\Omega \times X)$, $(K, \mathcal{K}, \mu, \Phi)$ is an invertible MDS on a standard probability space (for example, see [16, Theorem 2.8]), where \mathcal{K} is the σ -algebra generated by the Borel subset of K .

Proof of Theorem 1.2. Assume that $h_{\text{top}}(\phi, K) > 0$. By Proposition 2.6 there exists $\mu \in \mathcal{E}_{\mathbb{P}}^K(\Omega \times X)$ such that $h_\mu(\phi, K) > 0$. Since Φ is injective, $(K, \mathcal{K}, \mu, \Phi)$ is an invertible ergodic MDS on a standard probability space and $\pi_\Omega : (K, \mathcal{K}, \mu, \Phi) \rightarrow (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is a factor map between two MDSs on standard probability spaces. We divide the remainder of the proof into two steps.

Step 1. In this step, we obtain a ν -full subset of Y , which has some good properties to help us complete the proof of Theorem 1.2.

Let $\mathcal{P}_\mu(\pi_\Omega^{-1}\mathcal{F})$ be the relative Pinsker σ -algebra of $(K, \mathcal{K}, \mu, \Phi)$ with respect to $\pi_\Omega^{-1}\mathcal{F}$. By [13, Theorem 6.5 and Lemma 5.2], there exists an invertible Polish system (Y, \mathcal{Y}, ν, S) and two factor maps

$$\pi_1 : (K, \mathcal{K}, \mu, \Phi) \rightarrow (Y, \mathcal{Y}, \nu, S), \quad \pi_2 : (Y, \mathcal{Y}, \nu, S) \rightarrow (\Omega, \mathcal{F}, \mathbb{P}, \theta),$$

between invertible MDSs on standard probability spaces such that $\pi_2 \circ \pi_1 = \pi_\Omega$ and $\pi_1^{-1}(\mathcal{Y}) = \mathcal{P}_\mu(\pi_\Omega^{-1}\mathcal{F}) \pmod{\mu}$. That is, $\pi_1 : (K, \mathcal{K}, \mu, \Phi) \rightarrow (Y, \mathcal{Y}, \nu, S)$ is the relative Pinsker factor map with respect to $\pi_\Omega^{-1}\mathcal{F}$.

Denote d_Ω as the complete metric on Ω . Define the metric on $\Omega \times X$ as

$$\rho(\mathbf{k}_1, \mathbf{k}_2) = \max\{d_\Omega(\omega_1, \omega_2), d(x_1, x_2)\}$$

where $\mathbf{k}_i = (\omega_i, x_i) \in \Omega \times X$ for $i = 1, 2$. Then $(\Omega \times X, \rho)$ is a complete separable metric space.

Given $\delta > 0$ and $\mathbf{k}_0 \in K$, put

$$\mathcal{W}_a(K, \delta) = \{(\mathbf{k}_1, \mathbf{k}_2) \in K \times K : \limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N \rho((\Phi \times \Phi)^{a_i}(\mathbf{k}_1, \mathbf{k}_2)) > \delta\},$$

and

$$\mathcal{W}_a^s(\mathbf{k}_0, \Phi) = \{\mathbf{k} \in K : \lim_{n \rightarrow +\infty} \rho(\Phi^{an}\mathbf{k}, \Phi^{an}\mathbf{k}_0) = 0\}.$$

Let $\mu = \int_Y \mu_y d\nu(y)$ be the disintegration relative to the factor (Y, \mathcal{Y}, ν, S) . Claim that there is a ν -full measure set \hat{Y} such that for $y \in \hat{Y}$ with following properties:

(P1) Denoting $\omega = \pi_2(y)$, then $\phi(n, \omega) : X \rightarrow X$ is continuous and $K(\theta^n\omega)$ is a compact subset of X for each $n \in \mathbb{N}$.

(P2) $\mu_y(\pi_2(y) \times K_y) = 1$ and μ_y is non-atomic, where $K_y := \{x \in X : (\pi_2(y), x) \in K\}$.

(P3) There exists $\delta_y > 0$ such that $\mu_y \times \mu_y(\mathcal{W}_a(K, \delta_y)) = 1$.

(P4) For any $\mathbf{k} \in \pi_1^{-1}(y)$, one has that $\overline{\mathcal{W}_a^s(\mathbf{k}, \Phi)} \cap \text{supp}(\mu_y) = \text{supp}(\mu_y)$.

(P1) is the basic assumption. (P2) is due to $\mu(K) = 1$ and Lemma 2.8. Applying Theorem 3.1 and Theorem 3.8 to the factor map $\pi_\Omega : (K, \mathcal{K}, \mu, \Phi) \rightarrow (\Omega, \mathcal{F}, \mathbb{P}, \theta)$, respectively, (P3) and (P4) hold.

Step 2. In this step, we finish the proof of Theorem 1.2 by showing the following lemma.

Lemma 4.2. For any $y \in \hat{Y}$, there exists a Mycielski subset S_y of K_y and $\epsilon_0 > 0$ such that for any two distinct points $x_1, x_2 \in S_y$ satisfies (1.1) and (1.2).

Proof. Given $y \in \hat{Y}$, let

$$E_y := \{x \in K_y : \mu_y(\pi_2(y) \times U) > 0 \text{ for any open neighborhood } U \text{ of } x\}.$$

By (P2), we have that E_y is a perfect compact subset and

$$(4.1) \quad \pi_2(y) \times E_y = \text{supp}(\mu_y).$$

Given $\epsilon > 0$, let $\omega(y) := \pi_2(y) \in \Omega$,

$$\mathbf{P}(y) = \bigcap_{k=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{N=l}^{\infty} \left\{ (x_1, x_2) \in E_y \times E_y : \frac{1}{N} \sum_{j=1}^N d(\phi(a_j, \omega(y))x_1, \phi(a_j, \omega(y))x_2) < \frac{1}{k} \right\},$$

and

$$\mathbf{D}_\epsilon(y) = \bigcap_{k=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{N=l}^{\infty} \left\{ (x_1, x_2) \in E_y \times E_y : \frac{1}{N} \sum_{j=1}^N d(\phi(a_j, \omega(y))x_1, \phi(a_j, \omega(y))x_2) > \epsilon - \frac{1}{k} \right\}.$$

We complete the proof of this lemma by the following two claims.

Claim 4.3. Recall that δ_y is defined as (P3). Then $\mathbf{D}_{\delta_y}(y)$ is a dense subset of $(E_y \times E_y, d \times d)$.

Proof. For any non-empty open subsets U_1, U_2 of X with $(U_1 \times U_2) \cap (E_y \times E_y) \neq \emptyset$, one has

$$\begin{aligned} & \mu_y \times \mu_y \left(\left((\pi_2(y) \times (U_1 \cap E_y)) \times (\pi_2(y) \times (U_2 \cap E_y)) \right) \cap \mathcal{W}_{\mathbf{a}}(K, \delta_y) \right) \\ & \stackrel{\text{(P3)}}{=} \mu_y \times \mu_y \left((\pi_2(y) \times (U_1 \cap E_y)) \times (\pi_2(y) \times (U_2 \cap E_y)) \right) \\ & \stackrel{\text{(4.1)}}{=} \mu_y(\pi_2(y) \times (U_1 \cap E_y)) \cdot \mu_y(\pi_2(y) \times (U_2 \cap E_y)) > 0, \end{aligned}$$

which implies that

$$\left((\pi_2(y) \times (U_1 \cap E_y)) \times (\pi_2(y) \times (U_2 \cap E_y)) \right) \cap \mathcal{W}_{\mathbf{a}}(K, \delta_y) \neq \emptyset.$$

Hence there exist $x_j \in U_j \cap E_y$ such that $(\mathbf{k}_1, \mathbf{k}_2) \in \mathcal{W}_{\mathbf{a}}(K, \delta_y)$ where $\mathbf{k}_j = (\omega(y), x_j)$ for $j = 1, 2$. It follows that

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{j=1}^N d(\phi(a_j, \omega(y))x_1, \phi(a_j, \omega(y))x_2) = \limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{j=1}^N \rho(\Phi^{a_j}(\mathbf{k}_1), \Phi^{a_j}(\mathbf{k}_2)) > \delta_y.$$

Therefore, $\mathbf{D}_{\delta_y}(y)$ is dense in $E_y \times E_y$. □

Claim 4.4. $\mathbf{P}(y)$ is a dense subset of $(E_y \times E_y, d \times d)$.

Proof. By (P4), for any $\mathbf{k} \in \pi_1^{-1}(y)$ one has that

$$(4.2) \quad \overline{W_{\mathbf{a}}^s(\mathbf{k}, \Phi) \cap \text{supp}(\mu_y)} = \text{supp}(\mu_y),$$

Denoting

$$\text{Asy}_{\mathbf{a}}(K, \Phi) := \{(\mathbf{k}_1, \mathbf{k}_2) \in K \times K : \lim_{n \rightarrow \infty} \rho(\Phi^{a_n}(\mathbf{k}_1), \Phi^{a_n}(\mathbf{k}_2)) = 0\},$$

then $W_{\mathbf{a}}^s(\mathbf{k}, \Phi) \times W_{\mathbf{a}}^s(\mathbf{k}, \Phi) \subset \text{Asy}_{\mathbf{a}}(K, \Phi)$. Hence

$$\begin{aligned} \text{Asy}_{\mathbf{a}}(K, \Phi) \cap (\text{supp}(\mu_y) \times \text{supp}(\mu_y)) & \supset (W_{\mathbf{a}}^s(\mathbf{k}, \Phi) \times W_{\mathbf{a}}^s(\mathbf{k}, \Phi)) \cap (\text{supp}(\mu_y) \times \text{supp}(\mu_y)) \\ & \supset (W_{\mathbf{a}}^s(\mathbf{k}, \Phi) \cap \text{supp}(\mu_y)) \times (W_{\mathbf{a}}^s(\mathbf{k}, \Phi) \cap \text{supp}(\mu_y)). \end{aligned}$$

This combined with (4.2) implies that

$$(4.3) \quad \overline{\text{Asy}_{\mathbf{a}}(K, \Phi) \cap (\text{supp}(\mu_y) \times \text{supp}(\mu_y))} = \text{supp}(\mu_y) \times \text{supp}(\mu_y).$$

Denoting $\pi_X : \Omega \times X \rightarrow X$ as the projection, it is clear that

$$\pi_X \times \pi_X \left(\text{Asy}_{\mathbf{a}}(K, \Phi) \cap ((\pi_2(y) \times E_y) \times (\pi_2(y) \times E_y)) \right) \subset \mathbf{P}(y).$$

It follows that

$$\begin{aligned} \overline{\mathbf{P}(y)} & \supset \pi_X \times \pi_X \left(((\pi_2(y) \times E_y) \times (\pi_2(y) \times E_y)) \cap \overline{\text{Asy}_{\mathbf{a}}(K, \Phi)} \right) \\ & \stackrel{\text{(4.1)}}{=} \pi_X \times \pi_X \left((\text{supp}(\mu_y) \times \text{supp}(\mu_y)) \cap \overline{\text{Asy}_{\mathbf{a}}(K, \Phi)} \right) \\ & \supset \pi_X \times \pi_X \left((\text{supp}(\mu_y) \times \text{supp}(\mu_y)) \cap \text{Asy}_{\mathbf{a}}(K, \Phi) \right) \\ & \stackrel{\text{(4.3)}}{=} \pi_X \times \pi_X (\text{supp}(\mu_y) \times \text{supp}(\mu_y)) = E_y \times E_y. \end{aligned}$$

This shows that $\mathbf{P}(y)$ is a dense subset of $E_y \times E_y$. □

Denoting $\mathbf{C}(y) = \mathbf{P}(y) \cap \mathbf{D}_{\delta_y}(y)$, it is clear that $\mathbf{C}(y)$ is a dense G_δ subset of $E_y \times E_y$. According to Lemma 4.1, there exists a dense Mycielski subset $S_y \subseteq E_y$ such that

$$S_y \times S_y \subseteq \mathbf{C}(y) \cup \Delta_{E_y},$$

where $\Delta_{E_y} = \{(x, x) : x \in E_y\}$. Obviously, $S_y \subseteq E_y \subseteq K_y$ and if (x_1, x_2) is a pair of distinct points in S_y , then $(x_1, x_2) \in \mathbf{C}(y)$ which implies that

$$\begin{aligned} \liminf_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N d(\phi(a_i, \omega)x_1, \phi(a_i, \omega)x_2) &= 0, \\ \limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N d(\phi(a_i, \omega)x_1, \phi(a_i, \omega)x_2) &> \epsilon_0 := \delta_y/2. \end{aligned}$$

This finishes the proof of Lemma 4.2. \square

Since $\nu(\widehat{Y}) = 1$, there exists $\widehat{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\widehat{\Omega}) = 1$ such that for any $\omega \in \widehat{\Omega}$, one has that $\pi_2^{-1}(\omega) \cap \widehat{Y} \neq \emptyset$. Taking some $y \in \pi_2^{-1}(\omega) \cap \widehat{Y}$, then $S(\omega) := S_y$ is the desired subset. \square

4.2. Proof of Theorem 1.1. In this subsection, we use the natural extension of the measure-preserving systems to deal with that (X, ϕ) is not injective. The reason why this method is not applicable to a general infinite positive integer sequence \mathbf{a} is that the sequence \mathbf{a} will arise a deviation for (1.2). Note that the symbols in this subsection are independent of the symbols in Section 4.1.

Proof of Theorem 1.1. Assume that $h_{top}(\phi, K) > 0$. By Proposition 2.6 there exists $\mu \in \mathcal{E}_{\mathbb{P}}^K(\Omega \times X)$ such that $h_\mu(\phi, K) > 0$. $\pi_\Omega : (K, \mathcal{K}, \mu, \Phi) \rightarrow (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is a factor map between two MDS on standard probability spaces.

Let $\Pi_K : (\bar{K}, \bar{\mathcal{K}}, \bar{\mu}, \bar{\Phi}) \rightarrow (K, \mathcal{K}, \mu, \Phi)$ be the natural extension of $(K, \mathcal{K}, \mu, \Phi)$ and $\Pi_\Omega : (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\theta}) \rightarrow (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be the natural extension of $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. Define $\bar{\pi} : \bar{K} \rightarrow \bar{\Omega}$ by $\bar{\pi}((\omega_i, x_i)_{i \in \mathbb{Z}}) = (\omega_i)_{i \in \mathbb{Z}}$ for $(\omega_i, x_i)_{i \in \mathbb{Z}} \in \bar{K}$. Then

$$\bar{\pi} : (\bar{K}, \bar{\mathcal{K}}, \bar{\mu}, \bar{\Phi}) \rightarrow (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\theta})$$

is a factor map between two MDSs on standard probability spaces, where \bar{K} and $\bar{\Omega}$ are Borel subsets of the Polish spaces $(\Omega \times X)^{\mathbb{Z}}$ and $\Omega^{\mathbb{Z}}$, respectively, and $\pi_\Omega \circ \Pi_K = \Pi_\Omega \circ \bar{\pi}$.

We divide the remainder of the proof into three steps.

Step 1. In this step, we're going to introduce some notations for our proof.

By Lemma 2.3 and Lemma 2.7, we have

$$\begin{aligned} h_{\bar{\mu}}(\bar{\Phi}|\bar{\pi}) &= h_{\bar{\mu}}(\bar{\Phi}|\bar{\pi}) + h_{\bar{\mathbb{P}}}(\bar{\theta}|\Pi_\Omega) \\ &= h_{\bar{\mu}}(\bar{\Phi}|\Pi_\Omega \circ \bar{\pi}) = h_{\bar{\mu}}(\bar{\Phi}|\pi_\Omega \circ \Pi_K) \\ &= h_{\bar{\mu}}(\bar{\Phi}|\Pi_K) + h_{\mu}(\Phi|\pi_\Omega) = h_{\mu}(\Phi|\pi_\Omega) = h_{\mu}(\phi, K) > 0. \end{aligned}$$

Let $\mathcal{P}_{\bar{\mu}}(\bar{\pi}^{-1}\bar{\mathcal{F}})$ be the relative Pinsker σ -algebra of $(\bar{K}, \bar{\mathcal{K}}, \bar{\mu}, \bar{\Phi})$ with respect to $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\theta})$. Therefore, there exists a MDS (Y, \mathcal{Y}, ν, S) on a standard probability space and two factor maps

$$\pi_1 : (\bar{K}, \bar{\mathcal{K}}, \bar{\mu}, \bar{\Phi}) \rightarrow (Y, \mathcal{Y}, \nu, S), \quad \pi_2 : (Y, \mathcal{Y}, \nu, S) \rightarrow (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\theta})$$

between two MDSs on standard probability spaces such that $\pi_2 \circ \pi_1 = \bar{\pi}$ and $\pi_1^{-1}(\mathcal{Y}) = \mathcal{P}_{\bar{\mu}}(\bar{\pi}^{-1}\bar{\mathcal{F}}) \pmod{\bar{\mu}}$. That is, $\pi_1 : (\bar{K}, \bar{\mathcal{K}}, \bar{\mu}, \bar{\Phi}) \rightarrow (Y, \mathcal{Y}, \nu, S)$ is the relative Pinsker factor map with respect to $\bar{\pi}^{-1}\bar{\mathcal{F}}$.

Define metrics ρ , ρ_1 and ρ_2 on $X^{\mathbb{Z}}$, $\Omega^{\mathbb{Z}}$ and $(\Omega \times X)^{\mathbb{Z}}$, respectively as follows:

$$\rho(\vec{x}_1, \vec{x}_2) = \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \frac{d(x_i^1, x_i^2)}{1 + d(x_i^1, x_i^2)}, \quad \rho_1(\vec{\omega}_1, \vec{\omega}_2) = \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \frac{d_{\Omega}(\omega_i^1, \omega_i^2)}{1 + d_{\Omega}(\omega_i^1, \omega_i^2)}$$

$$\rho_2(\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2) = \max\{\rho(\vec{x}_1, \vec{x}_2), \rho_1(\vec{\omega}_1, \vec{\omega}_2)\},$$

where $\vec{x}_j = (x_i^j)_{i \in \mathbb{Z}} \in X^{\mathbb{Z}}$, $\vec{\omega}_j = (\omega_i^j)_{i \in \mathbb{Z}} \in \Omega^{\mathbb{Z}}$, $\vec{\mathbf{k}}_j = (\omega_i^j, x_i^j)_{i \in \mathbb{Z}} \in (\Omega \times X)^{\mathbb{Z}}$ for $j = 1, 2$, and d , d_{Ω} are the compatible complete metrics on X , Ω , respectively. For simplicity, we sometimes identify $(\Omega \times X)^{\mathbb{Z}}$ with $\Omega^{\mathbb{Z}} \times X^{\mathbb{Z}}$ by

$$((\omega_i, x_i))_{i \in \mathbb{Z}} \in (\Omega \times X)^{\mathbb{Z}} \simeq ((\omega_i)_{i \in \mathbb{Z}}, (x_i)_{i \in \mathbb{Z}}) \in \Omega^{\mathbb{Z}} \times X^{\mathbb{Z}}.$$

Step 2. In this step, we obtain a similar result as Lemma 4.2 for the natural extension system.

Given $\delta > 0$ and $\vec{\mathbf{k}}_0 \in \bar{K}$, put

$$\mathcal{W}(\bar{K}, \delta) = \{(\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2) \in \bar{K} \times \bar{K} : \limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N \rho_2((\bar{\Phi} \times \bar{\Phi})^i(\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2)) > \delta\},$$

and

$$W^s(\vec{\mathbf{k}}_0, \bar{\Phi}) = \{\vec{\mathbf{k}} \in \bar{K} : \lim_{n \rightarrow +\infty} \rho_2(\bar{\Phi}^n \vec{\mathbf{k}}, \bar{\Phi}^n \vec{\mathbf{k}}_0) = 0\}.$$

Let $\bar{\mu} = \int_Y \bar{\mu}_y d\nu(y)$ be the disintegration of $\bar{\mu}$ relative to the factor (Y, \mathcal{Y}, ν, S) . By Lemma 2.8, Theorem 3.8 and Theorem 3.1, there is a ν -full measure set \hat{Y} with $\nu(\hat{Y}) = 1$ such that for $y \in \hat{Y}$ with following properties:

- (P1) Denoting $\vec{\omega}(y) = (\omega_i(y))_{i \in \mathbb{Z}} := \pi_2(y)$, then $\phi(n, \omega_i) : X \rightarrow X$ is continuous and $K(\omega_i(y))$ is compact subset of X for each $n \in \mathbb{N}$ and $i \in \mathbb{Z}$.
- (P2) $\bar{\mu}_y(\pi_2(y) \times \bar{K}_y) = 1$ and $\bar{\mu}_y$ is non-atomic where $\bar{K}_y := \{\vec{x} \in \Pi_{i \in \mathbb{Z}} K(\omega_i(y)) : \phi(1, \omega_i)x_i = x_{i+1} \text{ for any } i \in \mathbb{Z}\}$.
- (P3) There exists $\delta_y > 0$ such that $\mu_y \times \mu_y(\mathcal{W}(\bar{K}, \delta_y)) = 1$.
- (P4) For any $\vec{\mathbf{k}} \in \bar{\pi}_1^{-1}(y)$, one has that $W^s(\vec{\mathbf{k}}, \bar{\Phi}) \cap \text{supp}(\bar{\mu}_y) = \text{supp}(\bar{\mu}_y)$.

With the similar argument as Lemma 4.2, we have the following lemma.

Lemma 4.5. For any $y \in \hat{Y}$, \bar{K}_y is a compact subset of $(X^{\mathbb{Z}}, \rho)$ and there exists a Mycielski subset \bar{S}_y of \bar{K}_y and $\epsilon_0 > 0$ such that for any distinct two points $\vec{x}_1, \vec{x}_2 \in \bar{S}_y$, one has

$$(4.4) \quad \liminf_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N \rho_2(\bar{\Phi}^i(\vec{\omega}(y), \vec{x}_1), \bar{\Phi}^i(\vec{\omega}(y), \vec{x}_2)) = 0,$$

$$(4.5) \quad \limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N \rho_2(\bar{\Phi}^i(\vec{\omega}(y), \vec{x}_1), \bar{\Phi}^i(\vec{\omega}(y), \vec{x}_2)) > \epsilon_0,$$

where $\vec{\omega}(y) = (\omega_i(y))_{i \in \mathbb{Z}} := \pi_2(y)$.

Step 3. In this step, we finish the proof of Theorem 1.1.

Recall that

$$\pi_2 : (Y, \mathcal{Y}, \nu, S) \rightarrow (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\theta}) \text{ and } \Pi_{\Omega} : (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\theta}) \rightarrow (\Omega, \mathcal{F}, \mathbb{P}, \theta),$$

where π_2 is a factor map between two MDSs and Π_{Ω} is the projection. Since $\nu(\hat{Y}) = 1$, by Lemma 3.14, there exists $\hat{\Omega} \in \bar{\mathcal{F}}$ with $\bar{\mathbb{P}}(\hat{\Omega}) = 1$ such that $(\Pi_{\Omega} \circ \pi_2)^{-1}(\omega) \cap \hat{Y} \neq \emptyset$ for each $\omega \in \hat{\Omega}$.

From now, fix $\omega \in \hat{\Omega}$ and $y \in \hat{Y}$ such that $\Pi_{\Omega} \circ \pi_2(y) = \omega$. Then we have $\omega = \omega_0(y)$. By Lemma 4.5, there exist $\epsilon_0 > 0$ and a Mycielski subset \bar{S}_y of \bar{K}_y such that for each pair (\vec{x}_1, \vec{x}_2)

of distinct points in \bar{S}_y , satisfies (4.4) and (4.5). Let $\eta : \bar{K}_y \rightarrow K_\omega$ be the natural projection of coordinate with $\eta(\vec{x}) = x_0$ for $\vec{x} = (x_i)_{i \in \mathbb{Z}} \in X^{\mathbb{Z}}$. Put $S_\omega = \eta(\bar{S}_y)$. Then $S_\omega \subseteq K_\omega$.

In the following we show that S_ω is a Mycielski chaotic set for (ω, ϕ) . Firstly we claim that map $\eta : \bar{S}_y \rightarrow S_\omega$ is injective. If this is not true, then there exist two distinct points \vec{x}_1, \vec{x}_2 in \bar{S}_y such that $\eta(\vec{x}_1) = \eta(\vec{x}_2)$, i.e. $x_0^1 = x_0^2$. Since $\vec{x}_1, \vec{x}_2 \in \bar{K}_y$, we have

$$(4.6) \quad x_i^1 = \phi(i, \omega_0(y))x_0^1 = \phi(i, \omega_0(y))x_0^2 = x_i^2$$

for each $i \in \mathbb{N}_0$. Thus

$$\lim_{n \rightarrow +\infty} \rho((\phi(n, \omega_i(y))x_i^1)_{i \in \mathbb{Z}}, (\phi(n, \omega_i(y))x_i^2)_{i \in \mathbb{Z}}) = \lim_{n \rightarrow +\infty} \rho((x_{i+n}^1)_{i \in \mathbb{Z}}, (x_{i+n}^2)_{i \in \mathbb{Z}}) \stackrel{(4.6)}{=} 0$$

which contradicts (4.4). Hence $\eta : \bar{S}_y \rightarrow S_\omega$ is injective. Since \bar{S}_y is a Mycielski set, $\bar{S}_y = \bigcup_{j \in \mathbb{N}} C_j$ where each C_j is a Cantor set. Since $\eta : (C_j, \rho) \rightarrow (\eta(C_j), d)$ is a one to one surjective continuous map and C_j is a compact subset of $(X^{\mathbb{Z}}, \rho)$, it follows that $\eta : C_j \rightarrow \eta(C_j)$ is a homeomorphism. Thus, $\eta(C_j)$ is a Cantor set. Hence that $S_\omega = \bigcup_{j \in \mathbb{N}} \eta(C_j)$ is a Mycielski set of K_ω .

Given a pair (x_1, x_2) of distinct points in S_ω , one has that

$$\begin{aligned} & \liminf_{N \rightarrow +\infty} \frac{1}{N} \sum_{j=1}^N d(\phi(j, \omega)x_1, \phi(j, \omega)x_2) \\ & \leq (1 + d(x_1, x_2)) \liminf_{N \rightarrow +\infty} \frac{1}{N} \sum_{j=1}^N \rho((\phi(j, \omega_i(y))x_i^1)_{i \in \mathbb{Z}}, (\phi(j, \omega_i(y))x_i^2)_{i \in \mathbb{Z}}) = 0, \end{aligned}$$

where $(x_i^k)_{i \in \mathbb{Z}} = \eta^{-1}(x_k)$ for $k = 1, 2$. This implies that (1.1) holds. On the other hand, recall that ϵ_0 is the constant in Lemma 4.5. Take $L \in \mathbb{N}$ such that $\frac{1}{2L} < \epsilon_0/6$. By

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=1}^N \rho((\phi(k, \omega_i(y))x_i^1)_{i \in \mathbb{Z}}, (\phi(k, \omega_i(y))x_i^2)_{i \in \mathbb{Z}}) \geq \epsilon_0,$$

there exists a sequence of positive integers $\{N_j\}_{j \in \mathbb{N}}$ with $L < N_1 < N_2 < \dots$ such that

$$\frac{1}{N_j} \sum_{k=1}^{N_j} \rho((\phi(k, \omega_i(y))x_i^1)_{i \in \mathbb{Z}}, (\phi(k, \omega_i(y))x_i^2)_{i \in \mathbb{Z}}) \geq \epsilon_0/2.$$

Note that for each $k \in \mathbb{N}$,

$$(\phi(k, \omega_i(y))x_i^1)_{i \in \mathbb{Z}} = (x_{k+i}^1)_{i \in \mathbb{Z}} \quad \text{and} \quad (\phi(k, \omega_i(y))x_i^2)_{i \in \mathbb{Z}} = (x_{k+i}^2)_{i \in \mathbb{Z}}.$$

It follows that

$$\frac{1}{N_j} \sum_{k=1}^{N_j} \sum_{|i| \leq L} \frac{d(x_{i+k}^1, x_{i+k}^2)}{2^i(1 + d(x_{i+k}^1, x_{i+k}^2))} \geq \frac{\epsilon_0}{2} - \frac{1}{N_j} \sum_{k=1}^{N_j} \sum_{|i| \geq L+1} \frac{d(x_{i+k}^1, x_{i+k}^2)}{2^i(1 + d(x_{i+k}^1, x_{i+k}^2))} \geq \frac{\epsilon_0}{2} - \frac{\epsilon_0}{3} = \frac{\epsilon_0}{6}.$$

For each $j \in \mathbb{N}$, there exists $i_j \in [-L, L]$ such that

$$\sum_{k=1}^{N_j} \frac{d(x_{i_j+k}^1, x_{i_j+k}^2)}{1 + d(x_{i_j+k}^1, x_{i_j+k}^2)} = \max_{|i| \leq L} \left\{ \sum_{k=1}^{N_j} \frac{d(x_{i+k}^1, x_{i+k}^2)}{1 + d(x_{i+k}^1, x_{i+k}^2)} \right\}.$$

Therefore,

$$\frac{1}{N_j} \sum_{k=1}^{N_j} d(x_{i_j+k}^1, x_{i_j+k}^2) \geq \frac{1}{N_j} \sum_{k=1}^{N_j} \frac{d(x_{i_j+k}^1, x_{i_j+k}^2)}{1 + d(x_{i_j+k}^1, x_{i_j+k}^2)}$$

$$\geq \frac{1}{3} \frac{1}{N_j} \sum_{k=1}^{N_j} \sum_{|i| \leq L} \frac{d(x_{i+k}^1, x_{i+k}^2)}{2^i(1 + d(x_{i+k}^1, x_{i+k}^2))} \geq \frac{\epsilon_0}{18},$$

which implies that

$$\begin{aligned} \limsup_{N \rightarrow \infty} \sum_{i=1}^N d(\phi(i, \omega)x_1, \phi(i, \omega)x_2) &\geq \limsup_{j \rightarrow \infty} \frac{1}{N_j + L} \sum_{i=1}^{N_j+L} d(\phi(i, \omega)x_1, \phi(i, \omega)x_2) \\ &\geq \limsup_{j \rightarrow \infty} \frac{N_j}{N_j + L} \frac{\epsilon_0}{18} > \frac{\epsilon_0}{36}. \end{aligned}$$

This finishes the proof of Theorem 1.1. \square

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