POSITIVE DENSITY SUBSETS IN AMENABLE GROUPS

CHUNLIN LIU

Abstract. For a countable discrete amenable group G, it turns out that for any subsets H of G and E of $\mathbb{Z}$ with positive densities, there exists $k \in \mathbb{N}$ which depends only on the densities of H and E such that $G^k \subset (H \cdot H^{-1})^{E-E}$.

1. Introduction

Since Furstenberg began using dynamical systems to study number theory, many well-known results in number theory were proved by ergodic theory such as Szemeredi's theorem, Hindman's theorem and so on (see for example [4] to learn about relevant contents). In classical number theory, one of the main themes of combinatorics is sum-product estimates. It goes back to Erdös and Szemerédi [1] who conjectured that for any finite subset A of $\mathbb{Z}$ (or $\mathbb{R}$), for every $\epsilon > 0$ one has

$$|A + A| + |A \cdot A| \gg |A|^{2-\epsilon}$$

where $A + A = \{a + b : a, b \in A\}$ and $A \cdot A = \{ab : a, b \in A\}$.

In [2], Fish raised a question:

Question 1.1. *For a given infinite set $E \subset \mathbb{Z}$, how much structure does the set $(E - E) \cdot (E - E)$ possess?*

Meanwhile, he used ergodic theory to study this question when E has positive density in $\mathbb{Z}$ and showed that given two subsets E_1, E_2 of $\mathbb{Z}$ with positive densities, there exists $k \in \mathbb{N}$ which depends only on the densities of E_1 and E_2 such that $k\mathbb{Z} \subset (E_1 - E_1) \cdot (E_2 - E_2)$, where $E_i - E_i = \{e - e' : e, e' \in E_i\}$, $i = 1, 2$.

As the research progressed, we began to wonder if there were similar results in larger groups. Thus, in this paper, we take advantage of the measure-preserving systems under amenable group actions to extend the above result of Fish to countable amenable groups as follows.

Theorem 1.2. *Let G be a countable amenable group. For any subsets H of G and E of $\mathbb{Z}$ with positive densities, there exists $k \in \mathbb{N}$ which depends only on the densities of H and E such that*

$$G^k \subset (H \cdot H^{-1})^{E-E}$$

where $G^k = \{g^k : g \in G\}$, $H \cdot H^{-1} = \{h(h')^{-1} : h, h' \in H\}$ and $(H \cdot H^{-1})^{E-E} = \{h^k : h \in H \cdot H^{-1}, k \in E - E\}$.

2020 *Mathematics Subject Classification.* Primary 37A44; Secondary 28D15, 11E25.

Key words and phrases. Amenable group, Positive density subgroup, Difference set.

Remark 1.3. In the proof of Theorem 1.2, we can obtain the exact value of $k \in \mathbb{N}$, which is in the form $k = (s+1)!(t^{(s+1)!}+1)!$, where $s, t \in \mathbb{N}$ depend only on the densities of E and H, respectively.

If we take $G = \mathbb{Z}$ or $\mathbb{Z}_N$, where $N \in \mathbb{N}$, then we obtain the results in [2]. However, we know that finite groups, solvable groups and finitely generated groups of subexponential growth are all amenable groups. In contrast to the special amenable group $\mathbb{Z}$, a general amenable group may have very complicated structure, which makes it harder to study. So we have more results by taking G as other groups rather than $\mathbb{Z}$. For instance a special case $G = \mathbb{Z}^d$, Theorem 1.2 implies that for every subset H of $\mathbb{Z}^d$ with positive density and every $m \in \mathbb{N}$ there exists $k \geq 1$ such that

$$k\mathbb{Z}^d \subset (m\mathbb{Z}) \cdot (E - E).$$

Moreover, if we let G be the Heisenberg group, that is, the two-step nilpotent countable matrix group

$$G = \left\{ \begin{pmatrix} 1 & m_3 & m_1 \\ 0 & 1 & m_2 \\ 0 & 0 & 1 \end{pmatrix} : m_1, m_2, m_3 \in \mathbb{Z} \right\},$$

then we may obtain some results about matrixes.

This paper is organized as follows. In Section 2, we recall some basic notions that we use in this paper. In Section 3, we prove the key lemma using ergodic theory, which is used to prove Theorem 1.2. In Section 4, we construct a system to prove Theorem 1.2.

2. Preliminaries

In this section, we recall some notations and concepts which are used later. The reader may see [6, Chapter 4] for more details.

2.1. Følner sequences.

A countable discrete group G is called amenable if there exists a sequence of non-empty finite subsets $\mathbf{F} = \{F_n\}_{n=1}^{\infty}$ of G such that

$$\lim_{n\to\infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0$$

holds for every $g \in G$ and such $\mathbf{F}$ is called a Følner sequence of G.

Let G be a countable infinite discrete amenable group and $\mathbf{F} = \{F_n\}_{n=1}^{\infty}$ be a Følner sequence of G. If H is a subset of G we write

$$\bar{d}_{\mathbf{F}}(H) = \limsup_{n\to\infty} \frac{|H \cap F_n|}{|F_n|}$$

and

$$d_{\mathbf{F}}(H) = \lim_{n\to\infty} \frac{|H \cap F_n|}{|F_n|}$$

if this limit exists. Then we define

$$d^*(H) = \sup_{\mathbf{F}} d_{\mathbf{F}}(H)$$

where the supremum is taken for all Følner sequences $\mathbf{F}$ of G such that $d_{\mathbf{F}}(H)$ exists. We remark that supremum is attained. We say H has positive density if $d^*(H) > 0$.

2.2. **Generic points.** In the following article, let G be a countable discrete group with the unit 1_G. By a G-system (X, G) we mean a compact metric space X endowed with a metric ρ, together with G acting on X by homeomorphism, that is, there exists a continuous map $\Psi : G \times X \to X, \Psi(g, x) = gx$ satisfying $\Psi(1_G, x) = x, \Psi(g_1, \Psi(g_2, x)) = \Psi(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and $x \in X$. Given a G-system (X, G), denote by $\mathcal{B}_X$ the collection of all Borel subsets of X and $M(X)$ the set of all Borel probability measures on X. For $\mu \in M(X)$, the support of μ is defined to be the set

$$supp(\mu) = \{x \in X : \mu(U) > 0 \text{ for every open neighborhood } U \text{ of } x\}.$$

It is clear that $supp(\mu)$ is a closed subset of X and $\mu(supp(\mu)) = 1$. $\mu \in M(X)$ is called G-invariant if $\mu(A) = \mu(g^{-1}A)$ for any $g \in G$ and $A \in \mathcal{B}_X$. Denote by $M(X, G)$ be the set of all G-invariant measures in $M(X)$. It is well known that if, in addition, G is amenable then $M(X, G) \neq \emptyset$ and $M(X, G)$ is a convex compact metric subspace of $M(X)$ under weak*-topology.

Given a countable discrete amenable group, let $\mathbf{F} = \{F_n\}_{n=1}^{\infty}$ be a Følner sequence of G and $\mu \in M(X, G)$. We say that $x_0 \in X$ is a generic point for μ along $\mathbf{F}$ if

$$\frac{1}{|F_n|} \sum_{g \in F_n} \delta_{gx_0} \to \mu \text{ weakly* as } n \to \infty,$$

where δ_x is the Dirac mass at x. This is equivalent to

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(gx_0) \to \int f d\mu$$

for each real-valued continuous function f on X. In this case, μ is G-invariant and supported on the closed orbit $\overline{orb(x_0, G)}$ of x_0 under G, where $orb(x_0, G) = \{gx_0 : g \in G\}$. By the definition of the generic points, one has the following result.

Lemma 2.1. *Let (X, G) be a G-system, $\boldsymbol{F} = \{F_n\}_{n=1}^{\infty}$ be a Følner sequence of G and $\mu \in M(X, G)$, where G is a countable discrete amenable group. If $x_0 \in X$ is a generic point for μ along $\boldsymbol{F}$, then*

$$d_{\boldsymbol{F}}(\{g \in G : gx_0 \in U\}) = \mu(U)$$

if U is clopen, that is, open and closed.

3. The key lemma

In this section, following ideas in [2], we prove our key lemma, which combined with Furstenberg correspondence principle [3] will allow us to prove Theorem 1.2.

Lemma 3.1. *Let (X,G) be a G-system with $\mu \in M(X,G)$. Given $A \in \mathcal{B}_X$ with $\mu(A) > 0$, for any $L \in \mathbb{N}$ and $g \in G$ there exists $1 \leq m \leq \lceil \frac{1}{\mu(A)^L} \rceil$ such that*

$$\{g^{lm}\}_{l=1}^{L} \subset R(A),$$

where $\lceil a \rceil := \min\{r \in \mathbb{Z} : r > a\}$ and $R(A) = \{g \in G : \mu(A \cap g^{-1}A) > 0\}$.

Proof. Given $g \in G$ and $L \in \mathbb{N}$, we consider the product system $Z = \prod_{l=1}^{L} X$ with the transformation $S = \prod_{l=1}^{L} g^l$, the product σ-algebra $\mathcal{B}_Z$ and the product measure $\nu = \prod_{l=1}^{L} \mu$. Then we obtain a $\mathbb{Z}$-system (Z,S) and the measurable subset $\tilde{A} = \prod_{l=1}^{L} A$ of Z with $\nu(\tilde{A}) = \mu(A)^L > 0$. By Poincaré's recurrence theorem (see for example [8, Page 26]) there exists $1 \leq m \leq \lceil \frac{1}{\mu(A)^L} \rceil$ such that $\nu(\tilde{A} \cap S^{-m}\tilde{A}) > 0$, that is, for any $l \in \{1, 2, \ldots, L\}$, we have

$$\mu(A \cap g^{-lm}A) > 0,$$

which implies that $g^{lm} \in R(A)$ for each $l \in \{1, 2, \ldots, L\}$. □

With the help of Lemma 3.1, we obtain the following amenable analogue of Theorem 1.1 in [2].

Theorem 3.2. *Let (X,G) be a G-system and (Y,T) be a $\mathbb{Z}$-system, where G is a countable discrete amenable group. Fix $\mu \in M(X,G)$ and $\nu \in M(Y,T)$. For any $A \in \mathcal{B}_X$ with $\mu(A) > 0$ and $B \in \mathcal{B}_Y$ with $\nu(B) > 0$, there exists $k \in \mathbb{N}$ depending only on $\mu(A)$ and $\nu(B)$ such that*

$$G^k \subset R(A)^{R(B)},$$

where $R(A)^{R(B)} = \{g^n : g \in R(A), n \in R(B)\}$.

Proof. Let $M = \lceil \frac{1}{\nu(B)} \rceil$. Then by Poincaré's recurrence theorem (see for example [8, Page 26]), for every $c \in \mathbb{Z} \backslash \{0\}$ there exist $1 \leq i < j \leq M$ such that

$$\nu\left((T^c)^{-i} B \cap (T^c)^{-j} B \right) > 0.$$

As ν is T-invariant, it follows that there exists $1 \leq r = r(c) \leq M$ $(r = j - i)$ such that $rc \in R(B)$.

Let $L = M!$, $N = \lceil \frac{1}{\mu(A)^L} \rceil$ and $k = L \cdot N!$. Then for any $g \in G$, by Lemma 3.1, there exists $1 \leq m_g \leq N$ such that

$$\{g^{lm_g}\}_{l=1}^{L} \subset R(A). \tag{3.1}$$

By the above choice of M and $\frac{k}{Lm_g} \in \mathbb{Z} \setminus \{0\}$, there exists $1 \leq t = t\left(\frac{k}{Lm_g}\right) \leq M$ such that $t \cdot \frac{k}{Lm_g} \in R(B)$. As $L/t \in \mathbb{N}$, one has $g^{\frac{Lm_g}{t}} \in R(A)$ by (3.1). Thus

$$g^k = (g^{\frac{Lm_g}{t}})^{t \cdot \frac{k}{Lm_g}} \in R(A)^{R(B)}.$$

Therefore, $G^k \subset R(A)^{R(B)}$, which completes the proof of Theorem 3.2. □

Remark 3.3. According to the proof of Theorem 3.2, for any $s, t \in \mathbb{N}$, we may choose $A \in \mathcal{B}_X$ with $\mu(A) = 1/t$ and $B \in \mathcal{B}_Y$ with $\nu(B) = 1/s$ such that $G^k \subset R(A)^{R(B)}$, where $k = (s+1)!(t^{(s+1)!}+1)!$. Indeed, we let $M = s+1$, $L = (s+1)!$ and $N = t^{(s+1)!}+1$.

4. Proof of Theorem 1.2

In this section, Let G be a countable discrete amenable group and H a subset of G with $d^*(H) > 0$. Following ideas of [3] we construct a G-system to prove Theorem 1.2 by Theorem 3.2.

4.1. **Construction of the system.** Given a countable discrete amenable group G with the unit 1_G, we construct a product space $\{0,1\}^G$. By definition, the product topology on $\{0,1\}^G$ is generated by the cylinder sets $\prod_{s\in G} A_s$ where each A_s is open and $A_s = \{0,1\}$ for all $s \in G$ outside of a finite subset of G. Every open set in $\{0,1\}^G$ is a countable union of such cylinder sets, which consequently generate the Borel σ-algebra. We define the action G on $\{0,1\}^G$ by $(sx)_t = x_{ts}$ for all $s, t \in G$ and $x \in \{0,1\}^G$. Given a subset H of G, we consider the indicator function $\mathbf{1}_H$ as an element of $\{0,1\}^G$ that we write as x_H, that is, $(x_H)_t = \mathbf{1}_H(t)$ for each $t \in G$. Then we define

(1) $X = \overline{\{gx_H : g \in G\}}$ is the closed orbit of x_H under G-action.

Let $A = \{x \in X : (x)_{1_G} = 1\}$ be the cylinder set. We have

(2) A is a clopen subset of X and $H = \{g \in G : gx_H \in A\}$.

Let $\mathbf{F}$ be a Følner sequence of G with $d_{\mathbf{F}}(H) > 0$. Replacing $\mathbf{F}$ by a subsequence we can assume that

(3) x_H is a generic point along $\mathbf{F}$ for some $\mu \in M(X, G)$.

Applying Lemma 2.1 on the clopen subset A of X, we have

(4) $\mu(A) = d_{\mathbf{F}}(H)$.

For any $g \in R(A)$, one has $\mu(A \cap g^{-1}A) > 0$. By (3), there exists $g_0 \in G$ such that $g_0 x_H \in A \cap g^{-1}A$, that is, $gg_0 x_H$, $g_0 x_H \in A$ and hence $gg_0, g_0 \in H$. Thus $g = (gg_0)g_0^{-1} \in H \cdot H^{-1}$. This implies that

(5) $R(A) \subset H \cdot H^{-1}$.

4.2. **Proof of Theorem 1.2.** By the above construction, we are able to prove our main result.

Proof of Theorem 1.2. Let G be a countable discrete amenable group. According to (5) in the above construction, for any subset H of G with positive density, there exists a G-system (X, G), $\mu \in M(X, G)$ and $A \in \mathcal{B}_X$ with $\mu(A) = d_{\mathbf{F}}(H) > 0$ along some Følner sequence $\mathbf{F}$ of G such that

$$R(A) \subset H \cdot H^{-1}.$$

Similarly, by Furstenberg correspondence principle [3], for any subset E of $\mathbb{Z}$ with positive density, there exists a $\mathbb{Z}$-system (Y, T), $\nu \in M(Y, T)$ and $B \in \mathcal{B}_Y$ with $\nu(B) = d_{\mathbf{F}'}(E) > 0$ along some Følner sequence $\mathbf{F}'$ of $\mathbb{Z}$ such that

$$R(B) \subset E - E.$$

By Theorem 3.2, there exists $k \in \mathbb{N}$ which depends only on the densities of H and E, such that $G^k \subset R(A)^{R(B)}$. Hence

$$G^k \subset (H \cdot H^{-1})^{E-E}.$$

This completes the proof of Theorem 1.2. □

Data Availability Statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Acknowledgement

C. Liu was partially supported by NNSF of China (12090012).

References

[1] P. Erdös and E. Szemerédi. On sums and products of integers. *Studies in pure mathematics.* Birkhäuser, Basel, (1983), 213-218.

[2] A. Fish. On product of difference sets for sets of positive density. *Proc. Amer. Math. Soc.* **146**(8), (2018), 3449-3453.

[3] H. Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. Analyse Math.* **31**, (1977), 204-256.

[4] H. Furstenberg. *Recurrence in ergodic theory and combinatorial number theory.* M. B. Porter Lectures. Princeton University Press, Princeton, N.J., 1981. xi+203 pp. ISBN: 0-691-08269-3.

[5] M. I. Kargapolov and J. I. Merzljakov. *Fundamentals of the Theory of Groups, Graduate Texts in Mathematic, 62.* Springer-Verlag, New York-Berlin, 1979. xvii+203 pp.

[6] D. Kerr and H. Li. *Ergodic Theory. Independence and Dichotomies (Springer Monographs in Mathematics).* Springer, Cham, 2016. xxxiv+431 pp.

[7] E. Lindenstrauss. Pointwise theorems for amenable groups. *Invent. Math.* **146**(2), (2001), 259-295.

[8] P. Walters. *An introduction to Ergodic Theory, Graduate Texts in Mathematic, 79.* Springer-Verlag, New York-Berlin, 1982. ix+250 pp.

C. Liu: CAS Wu Wen-Tsun Key Laboratory of Mathematics, School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026, PR China

E-mail address: `lcl666@mail.ustc.edu.cn`