RELATIVE SEQUENCE ENTROPY FOR AMENABLE GROUP ACTIONS

CHUNLIN LIU, CHANGLIN WANG AND WEISHENG WU

Abstract. We introduce the concept of relative sequence entropy for amenable group actions and explore the interplay between relative sequence entropy and Kronecker algebras for amenable group actions, and rigid algebras for abelian group actions. Our investigation extends to the application of relative sequence entropy in various mixing concepts within both measure-theoretic and topological systems. Additionally, we refine the notion of relative sequence entropy by introducing the concept of relative sequence entropy pairs for amenable group actions.

1. Introduction

A G-topological dynamical system (or G-system) is defined as a pair (X, G), where X represents a compact metric space, and G is a (semi)-group that acts continuously on X. If there exists a G-invariant Borel probability measure μ on X, it induces a measure-preserving system (or G-MPS) $(X, \mathcal{B}_X, \mu, G)$, where $\mathcal{B}_X$ is the Borel σ-algebra on X. There is a classical case where $G = \mathbb{Z}$, the additive group of integers. The standard $\mathbb{Z}$ action can be induced by a homeomorphism map $T : X \to X$, typically denoted as (X, T). In the following, we will always suppose that G is a discrete infinite countable amenable group and $\mathcal{F}$ is a Følner sequence in G (see Section 2 for details), unless specifically stated otherwise.

For $\mathbb{Z}$-MPS, sequence entropy was introduced by Kušnirenko [15] as an isomorphism invariant, who used it to distinguish between transformations with same entropy. In the same paper, Kušnirenko also demonstrated that a $\mathbb{Z}$-MPS has discrete spectrum if and only if its sequence entropy is zero along any sequence. From the view of Koopman–von Neumann [14], the opposite of discrete spectrum is weak mixing. Saleski [24] exhibited this property in the view of sequence entropy, namely, he proved a $\mathbb{Z}$-MPS is weakly mixing if and only if for any finite measurable partition, there exists an infinite sequence such that the sequence entropy with respect to this partition is positive. After that, Hulse [11] improved the Saleski's result, by finding a common sequence with respect to all finite measurable partitions. Furthermore, Hulse [12] used relative sequence entropy to distinguish weakly mixing extensions and compact extensions. Some results related to mild mixing and mildly mixing extensions can be found in [29, 30] by Zhang.

The relation between mixing, spectral properties and sequence entropy also can be studied from the following point of view. Let $h_\mu(X, T)$ be the measure-theoretical

Date: June 12, 2026.

2010 *Mathematics Subject Classification.* 37B40, 37B05, 37A05, 37A15.

Key words and phrases. Amenable group action, relative sequence entropy, Kronecker and rigid algebras, weakly mixing and compact extensions, weakly mixing set.

entropy of a $\mathbb{Z}$-MPS, and $\Pi(X)$ be its Pinsker σ-algebra, that corresponds its maximal zero entropy factor. A classical result in [23] is that given a finite measurable partition ξ,

$$\lim_{n\to\infty} h_\mu(T^n,\xi) = H_\mu(\xi|\Pi(X)).$$

This result can be reformulated using the language of measure-theoretical sequence entropy as follows:

$$\sup_{S=n\mathbb{Z}_+, n\in\mathbb{N}} h_\mu^S(T,\xi) = H_\mu(\xi|\Pi(X)).$$

When one considers the supremum over all increasing sequences S of positive integers, Huang, Maass, and Ye [8] proved that

$$\max_{S\subset\mathbb{Z}_+} h_\mu^S(T,\xi) = H_\mu(\xi|\mathcal{K}(X)), \tag{1}$$

where $\mathcal{K}(X)$ is the Kronecker algebra of (X,T). Based on the ideas of Huang, Maass, and Ye [8], Liu and Yan extended this result to amenable group actions [19], employing it to characterize weak mixing and discrete spectrum. Furthermore, this result was extended to general countable discrete infinite groups via a combinatorial method by Kerr and Li [13], and via using filters by Liu, Wang and Xu [18]. Coronel, Shao and Mass [2] showed that this result also holds for relative sequence entropy, and characterized compact, rigid, and mixing extensions via relative sequence entropy.

A natural question arises: can we characterize weakly mixing extensions and compact extensions for amenable group actions via some kind of entropy? Lott recently characterized weakly mixing extensions and compact extensions through relative slow entropy [20]. In this paper, we will show that sequence entropy can also be used to characterize those extensions for amenable group actions.

Firstly, we obtain a relative version of (1) under amenable group actions.

Theorem A. *Let $(X,\mathcal{B}_X,\mu,G)$ be a G-MPS, $(Y,\mathcal{B}_Y,\nu,G)$ its factor, and $\mathcal{F}$ a Følner sequence in G. Then for any sequence $D\subseteq G$ with positive upper density with respect to $\mathcal{F}$,*

$$\max_{S\subseteq D} h_\mu^{S,\mathcal{F}}(G,\xi|\mathcal{B}_Y) = H_\mu(\xi|\mathcal{K}(X|Y)).$$

The proof of Theorem A follows the lines of that of $\mathbb{Z}$-actions. However, we must overcome some difficulties, for example, the Birkhoff ergodic theorem under amenable group actions only holds along tempered Følner sequence [17].

Based on Theorem A, we employ the relative sequence entropy to characterize compact and weakly mixing extensions.

Corollary A.1. Let $(X,\mathcal{B}_X,\mu,G)$ be a G-MPS and $(Y,\mathcal{B}_Y,\nu,G)$ be its factor with factor mapping $\pi:(X,\mathcal{B}_X,\mu,G)\to(Y,\mathcal{B}_Y,\nu,G)$. Then

(1) π is a compact extension if and only if $h_\mu^{S,\mathcal{F}}(G|\mathcal{B}_Y)=0$ for any sequence $S\subseteq G$;
(2) π is a weakly mixing extension if and only if for any sequence D with positive upper density with respect to $\mathcal{F}$, there exists a subsequence $S\subseteq D$ such that $h_\mu^{S,\mathcal{F}}(G,\xi|\mathcal{B}_Y)=H_\mu(\xi|\mathcal{B}_Y)$ for any measurable partition ξ of X with $H_\mu(\xi|\mathcal{B}_Y)<\infty$.

Using Theorem A, Rokhlin Theorem and ideas in [2, Theorem 5.1] for $\mathbb{Z}$ actions, we also have the following interesting application. When $\mathcal{B}_Y$ is trivial, this result is obtained in [10, 19] for amenable groups and in [18] for general countable groups.

Corollary A.2. Suppose that $(Y, \mathcal{B}_Y, \nu, G)$ is a factor of an ergodic system $(X, \mathcal{B}_X, \mu, G)$. Then

$$\max_{S\subseteq G}\{h_\mu^{S,\mathcal{F}}(G|\mathcal{B}_Y)\} \in \{\log k : k \in \mathbb{N}\} \cup \{\infty\}.$$

Now we consider the rigid algebra, which is another important σ-algebra of a MPS. However, if G is nonabelian, the σ-algebra of rigid sets with respect to some sequence $\{g_n\}_{n=1}^\infty$ need not be G-invariant, and therefore the analogous reasoning of rigid algebra for $\mathbb{Z}$-actions breaks down. In this case one cannot infer the existence of a rigid factor. Nevertheless, we are able to characterize the mild mixing and rigid in the non-relative setting.

Theorem B. *Let $(X, \mathcal{B}_X, \mu, G)$ be a G-MPS. Then*

(1) $(X, \mathcal{B}_X, \mu, G)$ is mildly mixing if and only if for any $B \in \mathcal{B}_X$ and any infinite sequence F in G, there exists an infinite subsequence S of F such that $h_\mu^{S,\mathcal{F}}(G, \{B, B^c\}) > 0$;

(2) $(X, \mathcal{B}_X, \mu, G)$ is rigid if and only if there exists an infinite sequence F in G such that for any subsequence S of F, $h_\mu^{S,\mathcal{F}}(G) = 0$.

When G is an abelian group, we can consider the rigid extension rationally. Thus we will characterize the rigid extension via relative sequence entropy in this case as follows.

Theorem B'. Let $(X, \mathcal{B}_X, \mu, G)$ be a G-MPS and $(Y, \mathcal{B}_Y, \nu, G)$ its factor, where G is an Abelian group. Then

(1) $(X, \mathcal{B}_X, \mu, G)$ is a rigid extension of $(Y, \mathcal{B}_Y, \nu, G)$ if and only if there exists an IP-system $F = \{g_\alpha\}_{\alpha\in F(\mathbb{N})}$ such that $h_\mu^{S,\mathcal{F}}(G|\mathcal{B}_Y) = 0$ for any $F(\mathbb{N})$-monotone subsets $S \subseteq F$;

(2) $(X, \mathcal{B}_X, \mu, G)$ is a mild mixing extension of $(Y, \mathcal{B}_Y, \nu, G)$ if and only if any IP-system $F = \{g_\alpha\}_{\alpha\in F(\mathbb{N})}$, there exists an $F(\mathbb{N})$-monotone subset $S \subseteq F$ such that $h_\mu^{S,\mathcal{F}}(G, \xi|\mathcal{B}_Y) = H_\mu(\xi|\mathcal{B}_Y)$ for any measurable partition ξ of X with $H_\mu(\xi|\mathcal{B}_Y) < \infty$.

In the setting of topological dynamical systems, the topological sequence entropy also can be used to characterize various topological mixing. In 1974, Goodman [7] introduced the notion of topological sequence entropy and studied many properties of it. After that, Li [16] used it to characterize topologically weak mixing. Huang, Shao and Ye [9] further systematically investigated several topologically mixing concepts via topological sequence entropy. Recently, Liu and Yan [19] extended a part of aforementioned results to amenable group actions.

From a local viewpoint, Blanchard and Huang [1] defined a local version of weak mixing, so called weakly mixing set, and proved that positive topological entropy implies the existence of weakly mixing sets, which in particular, implies Li-Yorke chaos. Zhang and Oprocha showed under $\mathbb{Z}$ actions, a weakly mixing set has positive topological sequence entropy [21], and then they further showed that it has infinite topological sequence entropy [22]. In this paper, we will demonstrate that this result also holds for amenable group actions.

Theorem C. *Let (X, G) be a G-system. If A is a weakly mixing set, then the maximal topological sequence entropy of A is infinite, i.e. for any $m \in \mathbb{N}$, there is a sequence S such that $h_{top}^{S,\mathcal{F}}(G, A) \geq \log m$.*

Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems. Viewing each fiber $\pi^{-1}(y)$ as a subset of X, we define the relative topological sequence entropy. As a corollary of Theorem C, we have the following result.

Corollary C.1. Let (X, G) be a G-system and (Y, G) be its factor. If there exists a fiber $\pi^{-1}(y)$ which is a weakly mixing set in X, then the topological sequence entropy of (X, G) relative to (Y, G) is infinite.

The central concept in our paper hinges on the utilization of a Følner sequence to define relative sequence entropy, following a similar approach as presented by Liu and Yan (see [19]). The Birkhoff ergodic theorem for amenable group actions requires the use of a Følner sequence (see [17]), and in our paper, we need to utilize it as well. The reason for selecting a specific Følner sequence as the basis for defining relative sequence entropy is explained as follows:

(1) The notions of mixing and spectrum are defined by fixing a Følner sequence, and these definitions are independent of the specific choice of Følner sequence. For $\mathbb{Z}$ actions, $F_n = \{[0, n] \cap \mathbb{Z}\}_{n=1}^{\infty}$ forms a Følner sequence, and the definition of sequence entropy uses it.

(2) Our results will indicate to some extent that sequence entropy is independent of the choice of Følner sequences. Therefore, when studying certain properties, we may simply fix a simple Følner sequence to investigate them. This approach simplifies the analysis while still yielding meaningful results.

This paper is organized as follows. In Section 2, we will introduce the basic notions used in this paper. In Section 3, we will discuss modules and the relative spectrum. In Section 4, we will demonstrate the relation between relative measure-theoretical sequence entropy and weakly mixing extensions as well as compact extensions and prove Theorem A and Corollaries A.1 and A.2. In section 5, we consider the relation between measure-theoretical sequence entropy and rigid algebra, proving Theorems B and B'. In Section 6, we discuss relative topological sequence entropy and weakly mixing sets. Theorem C and Corollary C.1 are also proved there. Lastly, in Section 7, we will explore entropy pairs within the context of relative sequence entropy.

Acknowledgement. The authors would like to thank the referee for valuable suggestions. This work is supported by National Key R&D Program of China No. 2022YFA1007800 and NSFC Nos. 12071474, 12090012, 12031019 and 12090010.

2. Preliminaries

Throughout this paper, let X be a compact metric space equipped with a metric d, and let G be an infinite countable discrete group. A *G-system* refers to a pair (X, G), where $\Gamma : G \times X \to X, (g,x) \mapsto gx$, is a continuous mapping that satisfies the following conditions:

$$\Gamma(e_G, x) = x \text{ for each } x \in X;$$
$$\Gamma(g_1, \Gamma(g_2, x)) = \Gamma(g_1 g_2, x) \text{ for each } g_1, g_2 \in G \text{ and } x \in X.$$

Given a G-system (X, G), let $\mathcal{B}_X$ be the Borel σ-algebra of X, and $\mathcal{M}(X)$ the set of all Borel probability measures on X. We say that $\mu \in \mathcal{M}(X)$ is *G-invariant* if $\mu = g\mu := \mu \circ g^{-1}$ for each $g \in G$. Given a G-invariant probability measure μ, we call $(X, \mathcal{B}_X, \mu, G)$ a *measure-preserving system* (or G-MPS). A G-invariant measure μ is called *ergodic* if $\mu\left(\bigcup_{g\in G} gA\right) = 0$ or 1 for any $A \in \mathcal{B}_X$. However, for general G-systems, it is possible that there might not exist invariant measures. It is well known that the amenability of groups guarantees the existence of invariant probability measures.

Recall that an infinite countable discrete group G is called *amenable* if there exists a sequence $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ of finite subsets $F_n \subset G$ such that for any $g \in G$,

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0,$$

where $|\cdot|$ denotes the cardinality of a set, and $\triangle$ represents the symmetric difference of sets. The sequence $\mathcal{F}$ above-mentioned is called a *Følner sequence*. For convenience, we always assume that $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ is a Følner sequence of G such that $e_G \in F_1 \subseteq F_2 \subseteq \dots$ and $\cup_{n=1}^{\infty} F_n = G$.

2.1. **Measure decomposition.** We say that $(Y, \mathcal{B}_Y, \nu, G)$ is a *factor* of $(X, \mathcal{B}_X, \mu, G)$ if there exists a measurable map $\pi : X \to Y$ such that $\pi\mu = \nu$, and for any $g \in G$, $\pi \circ g = g \circ \pi$. Equivalently, one says that $(X, \mathcal{B}_X, \nu, G)$ is an *extension* of $(Y, \mathcal{B}_Y, \nu, G)$.

Let $(Y, \mathcal{B}_Y, \nu, G)$ be a factor of $(X, \mathcal{B}_X, \mu, G)$. There is an identification between $L^2(Y, \mathcal{B}_Y, \nu)$ and the subspace $L^2(X, \pi^{-1}(\mathcal{B}_Y), \mu) \subset L^2(X, \mathcal{B}_X, \mu)$ through $h \mapsto h \circ \pi$. Using this identification, we can define the projection of $L^2(X, \mathcal{B}_X, \mu)$ onto $L^2(Y, \mathcal{B}_Y, \nu)$ by the conditional expectation $f \mapsto E(f|\mathcal{B}_Y)$. Here the *conditional expectation* $E(f|\mathcal{B}_Y)$ is characterized as the unique $\mathcal{B}_Y$-measurable function in $L^2(Y, \mathcal{B}_Y, \nu)$ satisfying

$$\int_Y gE(f|\mathcal{B}_Y)d\nu = \int_X (g \circ \pi) f d\mu$$

for all $g \in L^2(Y, \mathcal{B}_Y, \nu)$.

The *disintegration* of μ over ν is represented by a measurable map $y \mapsto \mu_y$ from Y to the space of probability measures on X such that for any $f \in L^2(X, \mathcal{B}_X, \mu)$, we have

$$E(f|\mathcal{B}_Y)(y) = \int_X f d\mu_y \text{ for } \nu\text{-a.e. } y \in Y.$$

The *self-joining* of $(X, \mathcal{B}_X, \mu, G)$ relatively independent over the factor $(Y, \mathcal{B}_Y, \nu, G)$ is a G-MPS $(X \times X, \mathcal{B}_X \otimes \mathcal{B}_X, \mu \times {}_Y\mu, G)$, where the measure $\mu \times_Y \mu$ is defined by

$$\mu \times {}_Y\mu(B) = \int_Y \mu_y \times \mu_y(B) d\nu(y), \ \forall B \in \mathcal{B}_X \otimes \mathcal{B}_X.$$

From the definition, it is easy to see that for all $f_1, f_2 \in L^2(X, \mathcal{B}_X, \mu)$,

$$\int_{X\times X} f_1 \otimes f_2 \; d\mu \times_Y \mu = \int_Y E(f_1|\mathcal{B}_Y)E(f_2|\mathcal{B}_Y)d\nu,$$

where $f_1 \otimes f_2(x_1, x_2) = f_1(x_1)f_2(x_2)$. More details can be found in [3, 6].

2.2. **Measure-theoretical relative sequence entropy.** Let $(X, \mathcal{B}_X, \mu, G)$ be a G-MPS. Given a finite measurable partition ξ of X and a sub-σ-algebra $\mathcal{A}$ of $\mathcal{B}_X$, we define the conditional entropy of ξ given $\mathcal{A}$ as

$$H_\mu(\xi|\mathcal{A}) := \sum_{A\in\xi} \int_X -E(1_A|\mathcal{A}) \log E(1_A|\mathcal{A})\, \mathrm{d}\mu,$$

where $\mathbb{E}(1_A|\mathcal{A})$ denotes the conditional expectation of 1_A (i.e., the characteristic function of A) with respect to $\mathcal{A}$. It is well known that $H_\mu(\xi|\mathcal{A})$ increases with respect to ξ and decreases with respect to $\mathcal{A}$. Set $\mathcal{T} = \{\emptyset, X\}$ and define

$$H_\mu(\xi) := H_\mu(\xi|\mathcal{T}) = \sum_{A\in\xi} -\mu(A) \log \mu(A).$$

Definition 2.1. Let $S = \{s_n\}_{n\in\mathbb{N}} \subseteq G$ be an infinite sequence, and $\mathcal{F} = \{F_n\}$ be a Følner sequence. Then *sequence entropy of* (X, G) *relative to a sub-σ-algebra* $\mathcal{A}$ *with respect to a measurable partition* ξ *of* X *with* $H_\mu(\xi|\mathcal{A}) < \infty$, is defined by

$$h_\mu^{S,\mathcal{F}}(G, \xi|\mathcal{A}) = \limsup_{n\to\infty} \frac{1}{|F_n \cap S|} H_\mu(\bigvee_{g\in F_n\cap S} g^{-1}\xi|\mathcal{A}).$$

Then *sequence entropy of* (X, G) *relative to a sub-σ-algebra* $\mathcal{A}$ is defined by

$$h_\mu^{S,\mathcal{F}}(G|\mathcal{A}) = \sup_\xi\{h_\mu^{S,\mathcal{F}}(G, \xi|\mathcal{A}) : H_\mu(\xi|\mathcal{A}) < \infty\}.$$

Let $(Y, \mathcal{B}_Y, \nu, G)$ be a factor of $(X, \mathcal{B}_X, \mu, G)$ and $\{\mu_y\}_{y\in Y}$ be the disintegration of μ over ν. Then the conditional entropy of ξ relative to $\mathcal{B}_Y$ can be represented as

$$H_\mu(\xi|\mathcal{B}_Y) = \int_Y H_y(\xi)d\nu,$$

where $H_y(\cdot)$ denotes the entropy with respect to μ_y, and $\mathcal{B}_Y$ is viewed as a sub-σ-algebra of $\mathcal{B}_X$. The norms on Hilbert spaces $L^2(X, \mathcal{B}_X, \mu)$ and $L^2(X, \mathcal{B}_X, \mu_y)$, $y \in Y$ are denoted by $||\cdot||$ and $||\cdot||_y$, respectively.

Definition 2.2. A function $f \in L^2(X, \mathcal{B}_X, \mu)$ is called *almost periodic* over $\mathcal{B}_Y$ if for every $\epsilon > 0$ there exist $h_1, \dots, h_l \in L^2(X, \mathcal{B}_X, \mu)$ such that for all $g \in G$,

$$\min_{1\le j\le l} ||gf - h_j||_y < \epsilon, \text{for } \nu\text{-a.e. } y \in Y.$$

All almost periodic functions over $\mathcal{B}_Y$ constitute a subspace of $L^2(X, \mathcal{B}_X, \mu)$, denoted by $\overline{AP(\mathcal{B}_Y)}$. By a classical result in [31] (or [2, Proposition 2.1]), there exists a sub-σ-algebra $\mathcal{K}(X|Y)$ of $\mathcal{B}_X$ such that $\overline{AP(\mathcal{B}_Y)} = L^2(X, \mathcal{K}(X|Y), \mu)$, which is the so called *Kronecker algebra* of $(X, \mathcal{B}_X, \mu, G)$ relative to $(Y, \mathcal{B}_Y, \nu, G)$.

Definition 2.3. Let $(X, \mathcal{B}_X, \mu, G)$ and $(Y, \mathcal{B}_Y, \nu, G)$ be two G-MPSs with a factor map $\pi : (X, \mathcal{B}_X, \mu, G) \to (Y, \mathcal{B}_Y, \nu, G)$. Then
(1) π is called a *compact extension* if

$$L^2(X, \mathcal{K}(X|Y), \mu) = L^2(X, \mathcal{B}_X, \nu), \text{ i.e., } \mathcal{K}(X|Y) = \mathcal{B}_X;$$

(2) π is called a *weakly mixing extension* if $\mathcal{K}(X|Y) = \pi^{-1}(\mathcal{B}_Y)$.

To end this section, we recall some terminology from topological dynamical systems. A *family* is an upward hereditary collection $\mathcal{G}$ of subsets of G, i.e., a subset of G containing an element of $\mathcal{G}$ is in $\mathcal{G}$ too. If a family $\mathcal{G}$ is closed under finite intersections and satisfies $\emptyset \notin \mathcal{G}$, then it is called a *filter*. The *dual* of a family is $\mathcal{G}^* = \{F \subseteq G : F \cap F' \neq \emptyset \text{ for all } F' \in \mathcal{G}\}$. Let $\{x_n\}_{n\in G}$ be a sequence in a metric space (X, d). One says that x_n $\mathcal{G}$-converge to x, denoted by $\mathcal{G} - \lim x_n = x$, if for any neighborhood U of x, $\{n \in G : x_n \in U\} \in \mathcal{G}$.

Let $\mathcal{F} = \{F_n\}$ be a Følner sequence and A be a subset of G, the *upper density* of A with respect to $\mathcal{F}$ is

$$\overline{d}_{\mathcal{F}}(A) = \limsup_{n\to\infty} \frac{|A \cap F_n|}{|F_n|},$$

and the *lower density* $\underline{d}_{\mathcal{F}}$ is defined via replacing lim sup by lim inf. If $\overline{d}_{\mathcal{F}}(A) = \underline{d}_{\mathcal{F}}(A)$, we denote by $d_{\mathcal{F}}(A)$ the common value, and call the set A has density $d_{\mathcal{F}}(A)$ with respect to $\mathcal{F}$

3. Modules and relative spectrum

In this section, following ideas in [12], we obtain some results on modules, which will be used later. Let $(X, \mathcal{B}_X, \mu, G)$ be a G-MPS. Given a G-invariant sub-σ-algebra $\mathcal{A}$ of $\mathcal{B}_X$, there exist a G-MPS $(Y, \mathcal{B}_Y, \nu, G)$ and a factor map $\pi : (X, \mathcal{B}_X, \mu, G) \to (Y, \mathcal{B}_Y, \nu, G)$ such that $\pi^{-1}(\mathcal{B}_Y) = \mathcal{A}$. Let

$$\mu = \int \mu_y d\nu(y)$$

be the measure disintegration of μ with respect to $\mathcal{B}_Y$. Denote by $L^p(\mathcal{A})$ the set of all functions in $L^p(X, \mu)$ that are $\mathcal{A}$-measurable for $p \geq 1$. We denote the norm and inner product on $L^2(X, \mu_y)$ as $|\cdot|_y$ and $\langle \cdot, \cdot \rangle_y$, respectively.

Definition 3.1. A subset M of $L^2(X, \mu)$ is an $\mathcal{A}$-*module* if

$$f_1, f_2 \in M \implies h_1 f_1 + h_2 f_2 \in M,$$

for all $h_1, h_2 \in L^\infty(\mathcal{A})$. A basis of M is a set $\{f_i\} \subseteq M$ such that

$$M \subseteq \overline{\{\sum_i h_i f_i : h_i \in L^\infty(\mathcal{A}) \text{ for all } i\}}.$$

A module M is finite dimensional if it has a finite basis.

Let $M(\mathcal{A})$ denote the set of all functions in $L^2(X, \mu)$ that are contained in a finite-dimensional, G-invariant $\mathcal{A}$-module, and let $\overline{M}(\mathcal{A})$ denote its closure. The set $M(\mathcal{A})$ is closed under complex conjugation, and bounded functions in $M(\mathcal{A})$ form an algebra. Consequently, there exists a σ-algebra $\mathcal{D} \subseteq \mathcal{B}_X$ such that $\overline{M}(\mathcal{A}) = L^2(X, \mathcal{D}, \mu)$.

Furthermore, each $\mathcal{A}$-module M can be decomposed into closed linear subspaces of $L^2(X, \mu_y)$, denoted by M_y for ν-a.e. $y \in Y$ (see [12, Page 61] for more details). The following proposition can be proved by an argument similar to that of [12, Proposition 1]. However, we provide an alternative proof which demonstrates the finite dimension nature of $M(\mathcal{A})$.

Proposition 3.2. *Let $f \in L^2(X, \mu)$. Then $f \in \overline{M}(\mathcal{A})$ if and only if, for any $\epsilon > 0$ and infinite sequence $S = \{g_i\}_{i=1}^{\infty} \subseteq G$,*

$$\lim_{n\to\infty} \nu\left(\{y : \inf_{1\le j\le n-1} ||f \circ g_n - f \circ g_j||_y < \epsilon\}\right) = 1.$$

Proof. The "if" part follows from Remark 3.4 below. We prove "only if" part. Firstly, we have the following claim.

Claim. Given any infinite sequence $\{g_n\}_{n=1}^{\infty}$ of G, for ν-a.e. $y \in Y$, there exist a strictly increasing subsequence $\{n_i\}_{i=1}^{\infty}$ of $\mathbb{N}$ and $K_y > 0$ such that for any $i \in \mathbb{N}$,

$$\|f \circ g_{n_i}\|_y \le K_y.$$

Proof of Claim. By contradiction, there exists $A \in \mathcal{B}_Y$ with $\nu(A) > 0$ such that for any $y \in A$, $\liminf_{n\to\infty} \|f \circ g_n\|_y = \infty$. However,

$$\|f\|_2^2 = \lim_{n\to\infty} \|f \circ g_n\|_2^2 = \lim_{n\to\infty} \int_Y \|f \circ g_n\|_y^2 d\nu(y) \ge \int_A \liminf_{n\to\infty} \|f \circ g_n\|_y^2 d\nu(y) = \infty,$$

a contradiction against the assumption that $f \in L^2(X, \mu)$. $\square$

Assume the contrary that there exist $f \in \overline{M}(\mathcal{A})$, $\epsilon_0 > 0$ and an infinite sequence $S = \{g_n\}_{n=1}^{\infty} \subseteq G$ such that

$$\liminf_{n\to\infty} \nu(\{y : \inf_{1\le j\le n-1} ||f \circ g_n - f \circ g_j||_y < \epsilon_0\}) = 1 - \delta < 1.$$

Thus there exists a subsequence $\{n_i\}_{i=1}^{\infty}$ such that for any $i \in \mathbb{N}$,

$$\nu(\{y : \inf_{1\le j\le i-1} ||f \circ g_{n_i} - f \circ g_{n_j}||_y < \epsilon_0\}) \le 1 - 2\delta. \tag{2}$$

Denote by D the full set obtained by the above claim with respect to the infinite sequence $\{g_{n_i}\}_{i=1}^{\infty}$. For convenience, we still denote the subsequence by $\{g_{n_i}\}_{i=1}^{\infty}$, namely, for any $y \in D$, there exists $K_y > 0$ such that for any $i \in \mathbb{N}$,

$$\|f \circ g_{n_i}\|_y \le K_y. \tag{3}$$

Since $f \in \overline{M}(\mathcal{A})$, there exists a sequence $\{f_m\}_{m=1}^{\infty}$ in $M(\mathcal{A})$ such that

$$\lim_{m\to\infty} \|f_m - f\|_2 = 0.$$

Now we prove that for any $\epsilon > 0$ and $A \in \mathcal{B}_Y$ with $\nu(A) > 0$, there exist $y \in A$ and $J := J(y) > 0$ such that for any $m \ge J$,

$$\|f_m \circ g_{n_i} - f \circ g_{n_i}\|_y < \epsilon \text{ for any } i \in \mathbb{N}. \tag{4}$$

Indeed, by contradiction, there exist $\epsilon_1 > 0$ and $A_1 \in \mathcal{B}_Y$ with $\nu(A_1) > 0$, such that for any $y \in A_1$, there exist a strictly increasing sequence $\{m_j\}_{j=1}^{\infty}$ and a subsequence $\{n_{i_j}\}_{j=1}^{\infty}$ of $\{n_i\}_{i=1}^{\infty}$ such that

$$\|f_{m_j} \circ g_{n_{i_j}} - f \circ g_{n_{i_j}}\|_y > \epsilon_1.$$

Then by the G-invariance of μ, we have

$$\begin{aligned}\liminf_{j\to\infty} \|f_{m_j} - f\|_2^2 &= \liminf_{j\to\infty} \|f_{m_j} \circ g_{n_{i_j}} - f \circ g_{n_{i_j}}\|_2^2 \\ &\geq \liminf_{j\to\infty} \int_A \|f_{m_j} \circ g_{n_{i_j}} - f \circ g_{n_{i_j}}\|_y^2 d\nu(y) > \nu(A)\epsilon_1^2 > 0,\end{aligned}$$

which is a contradiction.

Let

$$E = D \cap \{y : \inf_{1\leq j\leq i-1} ||f \circ g_{n_i} - f \circ g_{n_j}||_y \geq \epsilon_0\}. \tag{5}$$

Then $\nu(E) > 0$ by (2), which implies that there exist $y \in E$ and $J := J(y) > 0$ such that for any $m \geq J$,

$$\|f_m \circ g_{n_i} - f \circ g_{n_i}\|_y < \epsilon_0/4, \text{ for any } i \in \mathbb{N}.$$

Furthermore, by (3), we have that for any $m \geq J$ and $i \in \mathbb{N}$,

$$\|f_m \circ g_{n_i}\|_y \leq \|f \circ g_{n_i}\|_y + \epsilon_0/4 \leq K_y + \epsilon_0/4.$$

Recall that for any $m \geq J$, $f_m \in M(\mathcal{A})$. Thus $\{f_m \circ g_{n_i}\}_{i=1}^{\infty}$ is a bounded infinite subset of some finite dimensional linear subspace M_y^m. Without loss of generality, by passing to a subsequence, we may assume that for each $m \in \mathbb{N}$, $\{f_m \circ g_{n_i}\}_{i=1}^{\infty}$ is a Cauchy sequence. However, we have proven that for any $m \geq J$,

$$\|f_m \circ g_{n_i} - f \circ g_{n_i}\|_y < \epsilon_0/4 \text{ and } \|f_m \circ g_{n_{i+1}} - f \circ g_{n_{i+1}}\|_y < \epsilon_0/4, \text{ for any } i \in \mathbb{N}.$$

Combining with (5), for any $i \in \mathbb{N}$,

$$\|f_m \circ g_{n_{i+1}} - f_m \circ g_{n_i}\|_y > \epsilon_0/2.$$

This is a contradiction against that $\{f_m \circ g_{n_i}\}_{i=1}^{\infty}$ is a Cauchy sequence. Thus we finish the proof of the "only if" part of the proposition. □

The following result is an extension of Hulse [12, Proposition 2] to amenable group actions. However, note that Birkhoff ergodic theorem is only available for tempered Følner sequence [17]. We do not provide details on tempered Følner sequence here, since it will not be used in the later proof. Our proof uses the von Neumann mean ergodic theorem for general Følner sequence to get L^1 convergence.

Proposition 3.3. *Let $f \in L^2(X, \mathcal{B}_X, \mu)$. Then $f \in \overline{M}(\mathcal{A})^{\perp}$ if and only if for all $h \in L^2(X, \mathcal{B}_X, \mu)$,*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} |E(h(f \circ g)|\mathcal{A})| = 0, \text{ in } L^1(\nu).$$

If in addition $\mathcal{F}$ is tempered, then these are equivalent to

$$\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{g\in F_n}|E(h(f\circ g)|\mathcal{A})|=0,\ \nu\text{-}a.e.$$

Proof. The "if" part follows from Remark 3.4 below. We prove "only if" part.

First we will show that if $f\in\overline{M}(\mathcal{A})^{\perp}$, then $E(f\otimes\bar{f}|\mathcal{I})=0$, $(\mu\times_Y\mu)$-a.e., where $\mathcal{I}$ is the σ-algebra generated by all G-invariant sets and $(f\otimes\bar{f})(x_1,x_2)=f(x_1)\bar{f}(x_2)$ for any $(x_1,x_2)\in X\times X$. For convenience, we denote $E(f\otimes\bar{f}|\mathcal{I})(x_1,x_2)$ by $E(x_1,x_2)$. Note that

$$\begin{aligned}\int|E|^2d\mu\times_Y\mu&=\int(f\otimes\bar{f})Ed\mu\times_Y\mu\\&=\int\left(\int\bar{f}(x)\left(\int E(x,w)f(w)d\mu_y(w)\right)d\mu_y(x)\right)d\nu(y).\end{aligned}$$

Since $f\in\overline{M}(\mathcal{A})^{\perp}$, we only need to prove that the function $x\mapsto\int E(x,w)f(w)d\mu_{\pi(x)}(w)$ is in $\overline{M}(\mathcal{A})$, which is equivalent to that $K(L^2(X,\mu))\subseteq\overline{M}(\mathcal{A})$, where K is the linear operator defined by

$$(Kf)(x)=\int E(x,w)f(w)d\mu_{\pi(x)}(w).$$

Consider the compact operators K_y on $L^2(X,\mu_y)$ defined by

$$K_yf=\int E(\cdot,w)f(w)d\mu_y(w).$$

Given any $y\in Y$, since K_y is a Hilbert-Schmidt operator, and $E(z,w)=\overline{E}(w,z)$, it follows that K_y is compact and symmetric. Applying the spectra theorem for compact, symmetric operators on K_y, we deduce that the eigenvalues of K_y form a bounded set of real numbers with no limit points except possibly 0. Furthermore, the eigenspaces span $L^2(X,\mu_y)$ and those eigenspaces corresponding to non-zero eigenvalues are finite dimensional. Let $\{\lambda_i(y_i)\}_{i\geq 1}$ be non-zero eigenvalues of K_y, indexed so that $\lambda_1\geq\lambda_2\geq\cdots$. Let $V_y^{(i)}$ be the finite-dimensional eigenspace corresponding to $\lambda_i(y)$ and let $V_y^{(0)}$ be the null space of K_y. (If K_y has only j non-zero eigenvalues, then $V_y^{(i)}=\{0\}$ for $i>j$). Note that if $K_yf=\lambda f$, μ_y-a.e. then for any $g\in G$, $(K_yf)\circ g^{-1}=\lambda f\circ g^{-1}$, μ_{gy}-a.e. However,

$$\begin{aligned}(K_yf)(g^{-1}x)&=\int E(g^{-1}x,w)f(w)d\mu_y(w)=\int E(g^{-1}x,g^{-1}w)f(g^{-1}w)d\mu_{gy}(w)\\&=\int E(x,w)f(g^{-1}w)d\mu_{gy}(w)=(K_{gy}(f\circ g^{-1}))(x).\end{aligned}$$

Then $K_{gy}(f\circ g^{-1})=\lambda(f\circ g^{-1})$. Therefore, we deduce that $K_yf=\lambda f$ if and only if $K_{gy}(f\circ g^{-1})=\lambda f\circ g^{-1}$ holds for all $g\in G$ and ν-a.e. $y\in Y$. Thus, there exists a one-to-one correspondence between the eigenvalues of K_y and K_{gy}, and the corresponding eigenspaces are isomorphic. Specifically, $\lambda_i(y)=\lambda_i(gy)$ for $i\geq 1$, and $g^{-1}V_y^{(i)}=V_{gy}^{(i)}$, where $g^{-1}V_y^{(i)}=\{f\circ g^{-1}:f\in V_y^{(i)}\}$, for $i\geq 0$. Furthermore, for each $i\geq 0$, $\lambda_i(y)$ is a measurable function of y, which shows that the set $\{y\in Y:$

$K_y f = \lambda_i(y) f\}$ is measurable for all $f \in L^2(X,\mu)$. Consequently, if denote $V^{(i)} = \{f \in L^2(X,\mu) : f \in V_y^{(i)}, \nu\text{-a.e.}\}$, then for $i \geq 1$, $V^{(i)}$ is K-invariant and is the direct sum of finite-dimensional, G-invariant $\mathcal{A}$-modules. $V^{(0)}$ represents the null space of K, and $L^2(X,\mu) = \sum_{i=0}^{\infty} V^{(i)}$. Hence, $K(L^2(X,\mu)) \subseteq \overline{M}(\mathcal{A})$, which implies $E(f \otimes \bar{f}|\mathcal{A}) = 0$, $(\mu \times_Y \mu)$-a.e.

By von Neumann mean ergodic theorem (see [6, Theorem 3.33] for example), we have that if $E(f \otimes \bar{f}|\mathcal{I}) = 0$, then

$$\frac{1}{|F_n|}\sum_{g\in F_n}(f \otimes \bar{f}) \circ g \to 0 \text{ in } L^2(X \times X, \mu \times_Y \mu).$$

Thus, for any $h \in L^2(X,\mu)$, one has that

$$\lim_{n\to\infty}\int \frac{1}{|F_n|}\sum_{g\in F_n}((f \otimes \bar{f}) \circ g) \cdot (h \otimes \bar{h}) d\mu \times_Y \mu = 0. \tag{6}$$

Note that

$$\begin{aligned}
&\int \frac{1}{|F_n|}\sum_{g\in F_n}((f \otimes \bar{f}) \circ g) \cdot (h \otimes \bar{h}) d\mu \times_Y \mu \\
=&\frac{1}{|F_n|}\sum_{g\in F_n}\int \left(\int (f \circ g) h d\mu_y\right)^2 d\nu(y) \\
\geq&\frac{1}{|F_n|}\sum_{g\in F_n}\left(\int |\int (f \circ g) h d\mu_y| d\nu(y)\right)^2,
\end{aligned}$$

which together with (6), implies that

$$\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{g\in F_n}\left(\int |\int (f \circ g) h d\mu_y| d\nu(y)\right)^2 = 0.$$

From standard analysis, this is equivalent to that

$$\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{g\in F_n}\int |\int (f \circ g) h d\mu_y| d\nu(y) = 0.$$

Thus we obtain that

$$\frac{1}{|F_n|}\sum_{g\in F_n}|E((f \circ g)h|\mathcal{A})| = \frac{1}{|F_n|}\sum_{g\in F_n}|\int (f \circ g) h d\mu_y| \to 0 \text{ in } L^1(Y,\nu).$$

If in addition $\mathcal{F}$ is tempered, then we can prove the second statement by a proof similar to that of [12, Proposition 2]. □

Remark 3.4. *Let $f = f_1 + f_2$ where $f_1 \in \overline{M}(\mathcal{A})$ and $f_2 \in \overline{M}(\mathcal{A})^{\perp}$.*

(1) If $f \notin \overline{M}(\mathcal{A})$, then f_2 is non-trivial, so there exists $\delta > 0$ such that $||f_2||_y^2 \geq \delta$ for y in a set E of positive measure ϵ. By the "only if" part of Proposition 3.3, we may choose a sequence $S = \{g_k\}_{k=1}^{\infty}$ such that

$$|\int f_2 \circ g_n \cdot f_2 \circ g_m d\mu_y| \leq \frac{1}{4}\delta$$

for all m, n and $y \in F$, where $\nu(F) > 1 - \frac{1}{2}\epsilon$. Then

$$||f_2 \circ g_n - f_2 \circ g_m||_y^2 \geq ||f_2 \circ g_n||_y^2 + ||f_2 \circ g_m||_y^2 - 2|\langle f_2 g_n, f_2 g_m \rangle_y| \geq \frac{1}{2}\delta$$

for $y \in g_n^{-1}E \cap F$, which is a set of measure greater than $\frac{1}{2}\epsilon$. Therefore, for all n,

$$\nu\left(\{y : \inf_{1 \leq i \leq n-1} ||f \circ g_n - f \circ g_i||_y^2 < \frac{\delta}{2}\}\right) < 1 - \frac{1}{2}\epsilon,$$

which is a contradiction. This proves the "if" part of Proposition 3.2.

(2) Now we prove the "if" part of Proposition 3.3. By the assumption, for any $h \in L^2(X, \mu)$,

$$0 = \limsup_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} |\int (f \circ g) h d\mu_y| = \limsup_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} |\int (f_1 \circ g) h d\mu_y|$$

in $L^1(X, \mu)$. By applying the argument in (1) to f_1 instead of f_2 there, we see that $f_1 \notin \overline{M}(\mathcal{A})$ which is a contradiction. This proves the "if" part of Proposition 3.3.

4. Kronecker algebra and relative sequence entropy

In this section, we establish the relation between relative sequence entropy and relative Kronecker algebra. We always assume that $(X, \mathcal{B}_X, \mu, G)$ is a G-MPS and $(Y, \mathcal{B}_Y, \nu, G)$ is its factor with the factor map π, and $\mathcal{A} = \pi^{-1}\mathcal{B}_Y$. Recall that the Kronecker algebra of $(X, \mathcal{B}_X, \mu, G)$ relative to $(Y, \mathcal{B}_Y, \nu, G)$ is defined in Definition 2.2 such that $\overline{AP(\mathcal{B}_Y)} = L^2(X, \mathcal{K}(X|Y), \mu)$. To simplify notation, denote $\mathcal{K}(X|Y)$ by $\mathcal{K}$.

Lemma 4.1. *Let ξ, η be measurable partitions such that $H_\mu(\xi|\mathcal{A}), H_\mu(\eta|\mathcal{A}) < \infty$. Then*

$$|h_\mu^{S,\mathcal{F}}(G, \xi|\mathcal{A}) - h_\mu^{S,\mathcal{F}}(G, \eta|\mathcal{A})| \leq \int H_y(\xi|\eta) + H_y(\eta|\xi) d\nu$$

for any infinite sequence S.

Proof. According to the classical entropy formula, we have for any $y \in Y$

$$H_y(\bigvee_{g \in F_n \cap S} g^{-1}(\xi \vee \eta)) = H_y(\bigvee_{g \in F_n \cap S} g^{-1}\xi) + H_y(\bigvee_{g \in F_n \cap S} g^{-1}\eta | \bigvee_{g \in F_n \cap S} g^{-1}\xi).$$

The same equation holds with ξ and η interchanged. Subtracting one from the other gives

$$\begin{aligned}
&|H_y(\bigvee_{g\in F_n\cap S} g^{-1}\xi) - H_y(\bigvee_{g\in F_n\cap S} g^{-1}\eta)| \\
=&|H_y(\bigvee_{g\in F_n\cap S} g^{-1}\eta| \bigvee_{g\in F_n\cap S} g^{-1}\xi) - H_y(\bigvee_{g\in F_n\cap S} g^{-1}\xi| \bigvee_{g\in F_n\cap S} g^{-1}\eta)| \\
\leq& H_y(\bigvee_{g\in F_n\cap S} g^{-1}\eta| \bigvee_{g\in F_n\cap S} g^{-1}\xi) + H_y(\bigvee_{g\in F_n\cap S} g^{-1}\xi| \bigvee_{g\in F_n\cap S} g^{-1}\eta) \\
\leq& \sum_{g\in F_n\cap S} (H_y(g^{-1}\xi|g^{-1}\eta) + H_y(g^{-1}\eta|g^{-1}\xi)).
\end{aligned}$$

Therefore, by the invariance of ν,

$$\begin{aligned}
&|\frac{1}{|F_n\cap S|}H_\mu(\bigvee_{g\in F_n\cap S} g^{-1}\xi|\mathcal{A}) - \frac{1}{|F_n\cap S|}H_\mu(\bigvee_{g\in F_n\cap S} g^{-1}\eta|\mathcal{A})| \\
\leq& \frac{1}{|F_n\cap S|}\int \sum_{g\in F_n\cap S} (H_y(g^{-1}\xi|g^{-1}\eta) + H_y(g^{-1}\eta|g^{-1}\xi))d\nu \\
=& \int (H_y(\xi|\eta) + H_y(\eta|\xi))d\nu.
\end{aligned}$$

The proof is completed. □

The following lemma was proved in [12, Lemma 2].

Lemma 4.2. *There exists a countable set $\{\xi_k\}_{k\in\mathbb{N}}$ of finite measurable partitions such that*

$$\inf_{k\in\mathbb{N}} \left\{ \int (H_y(\xi|\xi_k) + H_y(\xi_k|\xi))d\nu \right\} = 0$$

for all partitions ξ such that $H_\mu(\xi|\mathcal{A}) < \infty$.

Combining the above two lemmas, one has

$$h_\mu^{S,\mathcal{F}}(G|\mathcal{B}_Y) = \sup_\xi \{h_\mu^{S,\mathcal{F}}(G,\xi|\mathcal{B}_Y) : \xi \text{ is a finite measurable partition}\}.$$

Thus $h_\mu^{S,\mathcal{F}}(G|\mathcal{B}_Y) = 0$ if and only if $h_\mu^{S,\mathcal{F}}(G,\xi|\mathcal{B}_Y) = 0$ for all two-set measurable partitions ξ, since every finite partition is a refinement of two-set partitions. Therefore, in the following, we can only consider the finite measurable partitions.

Following ideas in [2, Lemma 3.6], we have the following result.

Lemma 4.3. *Let $\mathcal{F} = \{F_n\}_{n=1}^\infty$ be a Følner sequence and $D \subseteq G$ with $\overline{d}_\mathcal{F}(D) > 0$. Then there exists a subsequence $S \subseteq D$ such that*

$$h_\mu^{S,\mathcal{F}}(G,\xi|\mathcal{B}_Y) \geq H_\mu(\xi|\mathcal{K}(X|Y))$$

for any finite measurable partition ξ of X.

Proof. First, we prove the following claim:

Claim: Consider finite measurable partitions ξ and η of X. For any $\epsilon > 0$, there exist a sequence $D \subseteq G$ with $d_{\mathcal{F}}(D) = 1$ and $M \in \mathbb{N}$ such that for any $m > M$ and any $g \in (D \cap F_m)\backslash F_M$,

$$\int_Y H_y(g^{-1}\xi|\eta)d\nu \geq H_\mu(\xi|\mathcal{K}) - \epsilon.$$

Proof of the claim. Let $\xi = \{A_1, \ldots, A_k\}$ and $\eta = \{B_1, \ldots, B_l\}$. For any $A, B \in \mathcal{B}_X$, since $1_A - E(1_A|\mathcal{K}) \in L^2(X, \mathcal{K}, \mu)^\perp$, we have by Proposition 3.3,

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} \left| \int_X ((1_A - E(1_A|\mathcal{K})) \circ g) 1_B d\mu_y \right| = 0$$

in $L^1(Y, \mathcal{B}_Y, \nu)$.

Equivalently for any $\epsilon > 0$ there exist a sequence D' of G with $d_{\mathcal{F}}(D') = 1$ and $M' \in \mathbb{N}$ such that for any $g \in (D' \cap F_n)\backslash F_M$, $n > M'$, we have

$$\int_Y \left| \int_X ((1_A - E(1_A|\mathcal{K})) \circ g) 1_B d\mu_y \right| d\nu < \epsilon.$$

Moreover, there exist a sequence D with $d_{\mathcal{F}}(D) = 1$ and $M \in \mathbb{N}$ such that for any $g \in (D \cap F_n)\backslash F_M$, $n > M$,

$$\int_Y \left| \mu_y(g^{-1}A_i \cap B_j) - \int_X (E(1_{A_i}|\mathcal{K}) \circ g) 1_{B_j} d\mu_y \right| d\nu < \epsilon \text{ for any } 1 \leq i \leq k,\ 1 \leq j \leq l.$$

Let $\phi(x) = -x \log x$. For any $\gamma > 0$, we choose $0 < \delta < \frac{\gamma}{4}\frac{1}{kl}$ such that

$$|u - v| < \delta \to |\phi(u) - \phi(v)| < \frac{\gamma}{4}\frac{1}{kl}. \tag{7}$$

Taking $M \in \mathbb{N}$ large enough, for any $g \in (D \cap F_m)\backslash F_M, m > M$, there is $E_m \subseteq Y$ with $\nu(E_m) > 1 - \frac{\gamma}{2\log k}$ such that

$$\left| \mu_y(g^{-1}A_i \cap B_j) - \int_X (E(1_{A_i}|\mathcal{K}) \circ g) 1_{B_j} d\mu_y \right| < \delta \tag{8}$$

for all $1 \leq i \leq k$, $1 \leq j \leq l$ and $y \in E_m$.

Now let $a_{i,j} = -\left(\int_X (E(1_{A_i}|\mathcal{K}) \circ g) 1_{B_j} d\mu_y\right) \log \frac{\int_X (E(1_{A_i}|\mathcal{K})\circ g) 1_{B_j} d\mu_y}{\mu_y(B_j)}$. Then

$$a_{i,j} = \mu_y(B_j)\phi\left(\int_{B_j} (E(1_{A_i}|\mathcal{K}) \circ g) d\mu_{B_j,y}\right),$$

where $\mu_{B_j,y} = \mu(\cdot \cap B_j)/\mu_y(B_j)$. Since ϕ is concave, one deduces

$$\begin{aligned} a_{i,j} &\leq \mu_y(B_j) \int_{B_j} -(E(1_{A_i}|\mathcal{K}) \circ g) \log(E(1_{A_i}|\mathcal{K}) \circ g) d\mu_{B_j,y} \\ &= \int_{B_j} -(E(1_{A_i}|\mathcal{K}) \circ g) \log(E(1_{A_i}|\mathcal{K}) \circ g) d\mu_y. \end{aligned} \tag{9}$$

For any $y \in E_m$ and $g \in (D \cap F_m) \backslash F_M, m > M$, using (7) and (8), we have

$$\begin{aligned}
(10) \qquad & \left| \sum_{i,j} -\mu_y(g^{-1}A_i \cap B_j) \log \frac{\mu_y(g^{-1}A_i \cap B_j)}{\mu_y(B_j)} - a_{i,j} \right| \\
& \leq \sum_{i,j} \bigg| \mu_y(g^{-1}B_i \cap B_j) \log \mu_y(g^{-1}A_i \cap B_j) \\
& \quad - \int_X (E(1_{A_i}|\mathcal{K}) \circ g) 1_{B_j} d\mu_y \log \int_X (E(1_{A_i}|\mathcal{K}) \circ g) 1_{B_j} d\mu_y \bigg| \\
& \quad + \sum_i \left| \sum_j \left(\mu_y(g^{-1}A_i \cap B_j) - \int_X (E(1_{A_i}|\mathcal{K}) \circ g) 1_{B_j} d\mu_y \right) \log \mu_y(B_j) \right| \\
& \leq kl \frac{\gamma}{4} \frac{1}{kl} + k\delta \left| \log \left(\prod_j \mu_y(B_j) \right) \right| \\
(11) \qquad & \leq \frac{\gamma}{4} + k \frac{\gamma}{4} \frac{1}{kl \log l} \left| \log \left(\frac{\sum_j \mu_y(B_j)}{l} \right)^l \right| = \frac{\gamma}{2}.
\end{aligned}$$

Then using (10) and (11), we have

$$(12) \qquad H_y(g^{-1}\xi|\eta) = \sum_{i,j} -\mu_y(g^{-1}A_i \cap B_j) \log \frac{\mu_y(g^{-1}A_i \cap B_j)}{\mu_y(B_j)} \geq \sum_{i,j} a_{i,j} - \frac{\gamma}{2}.$$

Thus by (9) and (12),

$$H_y(g^{-1}\xi|\eta) \geq \sum_{i,j} a_{i,j} - \frac{\gamma}{2} \geq \sum_i \int_X -(E(1_{A_i}|\mathcal{K}) \circ g) \log(E(1_{A_i}|\mathcal{K}) \circ g) d\mu_y - \frac{\gamma}{2}.$$

Integrating with respect to ν one obtains

$$\begin{aligned}
& \int_Y H_y(g^{-1}\xi|\eta) d\nu \geq \int_{E_m} H_y(g^{-1}\xi|\eta) d\nu \\
& \geq \sum_i \int_{E_m} \int_X -(E(1_{A_i}|\mathcal{K}) \circ g) \log(E(1_{A_i}|\mathcal{K}) \circ g) d\mu_y d\nu(y) - \frac{\gamma}{2} \\
& \geq \sum_i \int_Y \int_X -(E(1_{A_i}|\mathcal{K}) \circ g) \log(E(1_{A_i}|\mathcal{K}) \circ g) d\mu_y d\nu(y) \\
& - \sum_i \int_{Y \backslash E_m} \int_X -(E(1_{A_i}|\mathcal{K}) \circ g) \log(E(1_{A_i}|\mathcal{K}) \circ g) d\mu_y d\nu(y) - \frac{\gamma}{2} \\
& \geq \sum_i \int_X -(E(1_{A_i}|\mathcal{K}) \circ g) \log(E(1_{A_i}|\mathcal{K}) \circ g) d\mu - \frac{\gamma}{2 \log k} \log k - \frac{\gamma}{2} \\
& = H_\mu(\xi|\mathcal{K}) - \gamma.
\end{aligned}$$

This completes the proof of the claim. □

Let $\{\xi_k\}_{k\in\mathbb{N}}$ be as in Lemma 4.2. For any sequence D' with $\overline{d}(D')>0$, by induction, there is an infinite subsequence $S=\{s_i\}_{i=1}^{\infty}\subseteq D'$ such that

$$\int_Y H_y(s_n^{-1}\xi_j|\bigvee_{i=1}^{n-1}s_i^{-1}\xi_j)d\nu \geq H_\mu(\xi_j|\mathcal{K})-\frac{1}{2^n},$$

for any $n\geq 2$ and $1\leq j\leq n$. Denote $F_n\cap S=\{s_{i_1},\dots,s_{i_{k_n}}\}$ is a subsequence of S. Fix $k\in\mathbb{N}$, one has

$$\begin{aligned}
&h_\mu^{S,\mathcal{F}}(G,\xi_k|\mathcal{B}_Y)\\
=&\limsup_{n\to\infty}\frac{1}{|F_n\cap S|}H_\mu(\bigvee_{g\in F_n\cap S}g^{-1}\xi_k|\mathcal{B}_Y)=\limsup_{n\to\infty}\frac{1}{|F_n\cap S|}\int_Y H_y(\bigvee_{g\in F_n\cap S}g^{-1}\xi_k)d\nu(y)\\
=&\limsup_{n\to\infty}\frac{1}{|F_n\cap S|}\int_Y[H_y(\bigvee_{j=1}^{k}s_{i_j}^{-1}\xi_k)+\sum_{j=k+1}^{k_n}H_y(s_{i_j}^{-1}\xi_k|\bigvee_{l=1}^{j-1}s_{i_l}^{-1}\xi_k)]d\nu\\
\geq&\limsup_{n\to\infty}\frac{1}{|F_n\cap S|}\int_Y[H_y(\bigvee_{j=1}^{k}s_{i_j}^{-1}\xi_k)+\sum_{j=k+1}^{k_n}H_y(s_{i_j}^{-1}\xi_k|\bigvee_{l=1}^{i_{j-1}}s_l^{-1}\xi_k)]d\nu\\
\geq&\limsup_{n\to\infty}\frac{1}{|F_n\cap S|}[\int_Y H_y(\bigvee_{j=1}^{k}s_j^{-1}\xi_k)d\nu+(k_n-k)H_\mu(\xi_k|\mathcal{K})-\sum_{i=k+1}^{k_n}\frac{1}{2^i}]=H_\mu(\xi_k|\mathcal{K}).
\end{aligned}$$

Therefore $h_\mu^{S,\mathcal{F}}(G,\xi_k|\mathcal{B}_Y)\geq H_\mu(\xi_k|\mathcal{K})$ for any $k\in\mathbb{N}$.

Now let ξ be any finite measurable partition. Given $\delta>0$, by Lemma4.1, choose ξ_k such that $\int_Y H_y(\xi|\xi_k)+H_y(\xi_k|\xi)d\nu(y)<\delta$. Then $|h_\mu^{S,\mathcal{F}}(G,\xi|\mathcal{B}_Y)-h_\mu^{S,\mathcal{F}}(G,\xi_k|\mathcal{B}_Y)|<\delta$ and $|H_\mu(\xi|\mathcal{B}_Y)-H_\mu(\xi_k|\mathcal{B}_Y)|<\delta$. So

$$h_\mu^{S,\mathcal{F}}(G,\xi|\mathcal{B}_Y)\geq h_\mu^{S,\mathcal{F}}(G,\xi_k|\mathcal{B}_Y)-\delta\geq H_\mu(\xi|\mathcal{B}_Y)-2\delta.$$

Since $\delta>0$ is arbitrary, the proof is completed. □

Lemma 4.4. *Given $B\in\mathcal{B}_X$, then $B\in\mathcal{K}(X|Y)$ if and only if $h_\mu^{S,\mathcal{F}}(G,\{B,B^c\}|\mathcal{B}_Y)=0$ for any infinite sequence $S\subseteq G$.*

Proof. The proof follows from similar ideas as in [12, Theorem 1].

($\Rightarrow$) Given $B\in\mathcal{K}(X|Y)$, denote $B_1=B$ and $B_2=B^c$ for convenience. Then $\xi=\{B_1,B_2\}$ is a finite measurable partition.

For any $n\in\mathbb{N}$, denote $F_n\cap S=\{g_1,\dots,g_{k_n}\}$. Then

$$\begin{aligned}
\frac{1}{|F_n\cap S|}H(\bigvee_{g\in F_n\cap S}g^{-1}\xi|\mathcal{B}_Y)&=\frac{1}{|F_n\cap S|}\int H_y(\bigvee_{g\in F_n\cap S}g^{-1}\xi)d\nu\\
&=\frac{1}{|F_n\cap S|}\int\sum_{r=1}^{k_n}H_y(g_r^{-1}\xi|\bigvee_{j=1}^{r-1}g_j^{-1}\xi)d\nu\\
&\leq\frac{1}{|F_n\cap S|}\sum_{r=1}^{k_n}\int\inf_{1\leq j\leq r}H_y(g_r^{-1}\xi|g_j^{-1}\xi)d\nu.
\end{aligned}\tag{13}$$

Let $\epsilon > 0$ and $\delta > 0$. By inequality (4) in the proof of Proposition 3.2., for $N \in \mathbb{N}$ large enough and $r \geq N$,

$$\inf_{1\leq j\leq r-1} ||1_{B_p} g_r - 1_{B_p} g_j||_y < \sqrt{\epsilon},$$

for $p = 1, 2$ on a subset E of Y with measure greater than $1 - \delta$. Note that

$$\|1_{B_p} \circ g_r - 1_{B_p} \circ g_j\|_y^2 = \mu_y(g_r^{-1} B_p \triangle g_j^{-1} B_p).$$

Hence for each $y \in E$ there exists j with $1 \leq j \leq n-1$ such that

$$|\mu_y(g_r^{-1} B_p \cap g_j^{-1} B_p) - \mu_y(g_j^{-1} B_p)| < \epsilon,$$

for $p = 1, 2$ and

$$\mu_y(g_r^{-1} B_p \cap g_j^{-1} B_q) < \epsilon,$$

for all $p, q = 1, 2$ and $p \neq q$. Now

$$H_y(g_r^{-1}\xi | g_j^{-1}\xi) = \sum_q \phi(\mu_y(g_r^{-1} B_q \cap g_j^{-1} B_q) - \phi(\mu_y(g_j^{-1} B_q))) + \sum_{p\neq q} \phi(\mu_y(g_r^{-1} B_p \cap g_j^{-1} B_q)),$$

where $\phi(w) = -w \log w$. By the continuity, we may choose $\epsilon > 0$ small enough such that $|w_1 - w_2| < \epsilon$ implies $|\phi(w_1) - \phi(w_2)| < \epsilon'$. Then $H_y(g_r^{-1}\xi | g_j^{-1}\xi) < 4\epsilon'$ for all $y \in E$. Note that $H_y(g_r^{-1}\xi | g_j^{-1}\xi) < \log 2$ for all $y \in Y$, so

$$\int \inf_{1\leq j\leq r} H_y(g_r^{-1}\xi | g_j^{-1}\xi) d\nu < (1 - \delta)4\epsilon' + \delta \log 2.$$

Since ϵ' and δ are arbitrary, by (13), we deduce that

$$\lim_{n\to\infty} \frac{1}{|F_n \cap S|} H_\mu\Big(\bigvee_{g\in F_n\cap S} g^{-1}\xi | \mathcal{B}_Y\Big) = 0.$$

($\Leftarrow$) By contradiction, we assume that there exists $B \notin \mathcal{K}(X|Y)$ such that

$$h_\mu^{S,\mathcal{F}}(G, \{B, B^c\} | \mathcal{B}_Y) = 0$$

for any infinite sequence $S \subseteq G$. However, by Lemma 4.3, there exists a sequence S such that

$$h_\mu^{S,\mathcal{F}}(G, \{B, B^c\} | \mathcal{B}_Y) \geq H_\mu(\{B, B^c\} | \mathcal{K}(X|Y)) > 0,$$

which is a contradiction. Now we finish the proof. □

Lemma 4.5. *For any finite measurable partition ξ of X and any infinite sequence $S \subseteq G$,*

$$h_\mu^{S,\mathcal{F}}(G, \xi | \mathcal{B}_Y) \leq H_\mu(\xi | \mathcal{K}(X|Y)).$$

Proof. Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a countable set of finite $\mathcal{K}(X|Y)$-measurable partitions such that $\{\xi_k\} \nearrow \mathcal{K}(X|Y)$. Since ξ_k is $\mathcal{K}(X|Y)$-measurable, by Lemma 4.4 one has $h_\mu^{S,\mathcal{F}}(G, \xi_k | \mathcal{B}_Y) =$

0. So

$$\begin{aligned}
h_\mu^{S,\mathcal{F}}(G,\xi|\mathcal{B}_Y) &= \limsup_{n\to\infty}\frac{1}{|F_n\cap S|}\int_Y H_y(\bigvee_{g\in F_n\cap S} g^{-1}\xi)d\nu - h_\mu^{S,\mathcal{F}}(G,\xi_k|\mathcal{B}_Y)\\
&\le \limsup_{n\to\infty}\frac{1}{|F_n\cap S|}\int_Y H_y(\bigvee_{g\in F_n\cap S} g^{-1}(\xi\vee\xi_k)) - H_y(\bigvee_{g\in F_n\cap S} g^{-1}\xi_k)d\nu\\
&= \limsup_{n\to\infty}\frac{1}{|F_n\cap S|}\int_Y H_y(\bigvee_{g\in F_n\cap S} g^{-1}\xi|\bigvee_{g\in F_n\cap S} g^{-1}\xi_k)d\nu\\
&\le \limsup_{n\to\infty}\frac{1}{|F_n\cap S|}\int_Y |F_n\cap S| H_y(\xi|\xi_k)d\nu\\
&= \int_Y H_y(\xi|\xi_k)d\nu = H_\mu(\xi|\xi_k\vee\mathcal{B}_Y)\le H_\mu(\xi|\xi_k).
\end{aligned}$$

The proof is completed by martingale theorem. □

Combining Lemmas 4.3 and 4.5, we finish the proof of Theorem A, namely, for any $D\subseteq G$ with $\overline{d}(D)>0$,

$$\max_{S\subseteq D} h_\mu^{S,\mathcal{F}}(G,\xi|\mathcal{B}_Y) = H_\mu(\xi|\mathcal{K}(X|Y)).$$

As a consequence, we obtain Corollary A.1, namely, the characterization of compact extensions and weakly mixing extensions (see Definition 2.3) via the relative sequence entropy:

Corollary 4.6. *Let $(X,\mathcal{B}_X,\mu,G)$ be G-MPS and $(Y,\mathcal{B}_Y,\nu,G)$ be its factor.*

(1) $(X,\mathcal{B}_X,\mu,G)$ is a compact extension of $(Y,\mathcal{B}_Y,\nu,G)$ if and only if $h_\mu^{S,\mathcal{F}}(G|\mathcal{B}_Y)=0$ for any sequence $S\subseteq G$;

(2) $(X,\mathcal{B}_X,\mu,G)$ is a weakly mixing extension of $(Y,\mathcal{B}_Y,\nu,G)$ if and only if for any sequence D with positive upper density with respect to $\mathcal{F}$, there exists a subsequence $S\subseteq D$ such that $h_\mu^{S,\mathcal{F}}(G,\xi|\mathcal{B}_Y)=H_\mu(\xi|\mathcal{B}_Y)$ for any measurable partition ξ of X with $H_\mu(\xi|\mathcal{B}_Y)<\infty$.

Following ideas in [2, Theorem 5.1], by Rokhlin Theorem and Theorem A, we obtain Corollary A.2:

Theorem 4.7. *Suppose $(Y,\mathcal{B}_Y,\nu,G)$ is a factor of the ergodic system $(X,\mathcal{B}_X,\mu,G)$. Then*

$$\max_{S\subseteq G}\{h_\mu^{S,\mathcal{F}}(G|\mathcal{B}_Y)\}\in\{\log k: k\in\mathbb{N}\}\cup\{\infty\}.$$

Remark 4.8. *If μ is not ergodic, then this result is incorrect even in the non-relative setting. In particular, there exists a minimal system such that, for any nonnegative real number t, there exists an invariant measure whose maximal sequence entropy equals to t (see* [18, Proposition 2.10]*).*

5. Measure-theoretical rigid and mild mixing

5.1. **Rigid and mild mixing for non-relative case.** In this section, following ideas in the work of Huang, Shao and Ye [9], we characterize rigid and mild mixing via sequence entropy. Let us begin with some definitions.

Definition 5.1 (see[26] or [25])**.** Let $(X, \mathcal{B}_X, \mu, G)$ be a G-MPS. Then it is called

(1) rigid if there exists an infinite sequence $\{g_n\}_{n=1}^{\infty}$ in G such that for any $A \in \mathcal{B}_X$,
$$\lim_{n\to\infty} \mu(A\Delta g_n^{-1}A) = 0;$$
(2) mild mixing if for any infinite sequence $\{g_i\}_{i=1}^{\infty}$ of G,
$$\liminf_{i\to\infty} \mu(B\Delta g_i^{-1}B) > 0, \text{ for any } B \in \mathcal{B}_X \text{ with } 0 < \mu(B) < 1.$$

If G is an abelian group, for any $A \in \mathcal{B}_X$ being rigid with respect to a sequence $\{g_n\}_{n=1}^{\infty}$, i.e., $\lim_{n\to\infty} \mu(A\Delta g_n^{-1}A) = 0$, then so is gA as is seen in the following calculation
$$\lim_{n\to\infty} \mu\left(g_n(gA)\Delta(gA)\right) = \lim_{n\to\infty} \mu\left(gg_nA\Delta gA\right) = \lim_{n\to\infty} \mu\left(g_nA\Delta A\right) = 0.$$
However, if G is nonabelian, this reasoning breaks down, and the σ-algebra of rigid sets with respect to some sequence $\{g_n\}_{n=1}^{\infty}$ need not be G-invariant. In this case one cannot infer the existence of a rigid factor. Nevertheless, we can characterize mild mixing and rigidity via sequence entropy for amenable group actions, without using the notion of rigid σ-algebra. We do it in the non-relative setting, since the relative case is much more involved.

Since the Hilbert space $L^2(X, \mu)$ is separable, for any infinite sequence in G, there exists an infinite subsequence $S = \{g_n\}_{n=1}^{\infty}$ of G such that $\lim_{n\to\infty}\langle \varphi, U_{g_n}\psi\rangle$ exists for any $\varphi, \psi \in L^2(X, \mu)$. Now for a fixed $\psi \in L^2(X, \mu)$, we define $J : L^2(X, \mu) \to \mathbb{C}$ by
$$J(\varphi) = \lim_{n\to\infty}\langle \varphi, U_{g_n}\psi\rangle.$$
Obviously, J is a continuous linear functional on $L^2(X, \mu)$. By the Riesz representation theorem, there exists $S(\psi) \in L^2(X, \mu)$ such that $J(\varphi) = \langle \varphi, S(\psi)\rangle$ for all $\varphi \in L^2(X, \mu)$. Note that if $\psi \geq 0$, then $S(\psi) \geq 0$. Indeed, suppose for contradiction that $\mu(\{x \in X : S(\psi)(x) < 0\}) > 0$. Then, it follows that
$$\langle 1_{\{x\in X:S(\psi)(x)<0\}}, S(\psi)\rangle < 0.$$
However, on the other hand, as $\psi \geq 0$, one has
$$\langle 1_{\{x\in X:S(\psi)(x)<0\}}, S(\psi)\rangle = \lim_{n\to\infty}\langle 1_{\{x\in X:S(\psi)(x)<0\}}, U_{g_n}\psi\rangle \geq 0,$$
which yields a contradiction.

Lemma 5.2. *Let $(X, \mathcal{B}_X, \mu, G)$ be a G-MPS, and $S = \{g_n\}_{n=1}^{\infty}$ be an infinite subsequence of G as above. Then for any finite measurable partition α of X, there exists an infinite sequence $\tilde{S} = \{\tilde{g}_i\}_{i=1}^{\infty}$ of S such that*
$$h_\mu^{\tilde{S},\mathcal{F}}(G, \alpha) \geq \sum_{B\in\alpha}\int_X -S(1_B)\log S(1_B)d\mu.$$

Proof. Enumerate α as $\{B_1,\dots,B_k\}$, and fix any finite measurable partition $\eta=\{D_1,\dots D_l\}$. Then

$$\lim_{i\to\infty}\langle U_{g_i}1_{B_s},1_{D_t}\rangle=\langle 1_{D_t},S(1_{B_s})\rangle$$

for any $s\in\{1,2,\dots,k\}$ and $t\in\{1,2,\dots,l\}$. Therefore we deduce

$$\begin{aligned}
\liminf_{i\to\infty}H_\mu(g_i^{-1}\alpha|\eta)&=\liminf_{i\to\infty}\sum_{s,t}-\mu(g_i^{-1}B_s\cap D_t)\log\left(\frac{\mu(g_i^{-1}B_s\cap D_t)}{\mu(D_t)}\right)\\
&=\lim_{i\to\infty}\sum_{s,t}-\langle U_{g_i}1_{B_s},1_{D_t}\rangle\log\left(\frac{\langle U_{g_i}1_{B_s},1_{D_t}\rangle}{\mu(D_t)}\right)\\
&=\sum_{s,t}-\langle S(1_{B_s}),1_{D_t}\rangle\log\left(\frac{\langle S(1_{B_s}),1_{D_t}\rangle}{\mu(D_t)}\right)\\
&=\sum_{s,t}-\mu(D_t)\left(\int_{D_t}\frac{S(1_{B_s})}{\mu(D_t)}d\mu\right)\log\left(\int_{D_t}\frac{S(1_{B_s})}{\mu(D_t)}d\mu\right)\\
&\geq\sum_{s,t}-\mu(D_t)\int_{D_t}\frac{S(1_{B_s})}{\mu(D_t)}\log\left(S(1_{B_s})\right)d\mu\\
&=\sum_{B\in\alpha}-\int_X S(1_B)\log\left(S(1_B)\right)d\mu,
\end{aligned}\tag{14}$$

where in the inequality we use concavity of the function $-x\log x$.

Now we can define inductively an infinite subsequence $\tilde{S}=\{\tilde{g}_i\}_{i=1}^\infty$ of S by using (14) repeatedly such that for any $i\in\mathbb{N}$,

$$H_\mu\left(\tilde{g}_i^{-1}\alpha|\vee_{j=1}^{i-1}\tilde{g}_j^{-1}\alpha\right)\geq\sum_{B\in\alpha}-\int_X S(1_B)\log\left(S(1_B)\right)d\mu-\frac{1}{2^i}.$$

For each $n\in\mathbb{N}$, there exist $1\leq i_1<i_2<\ldots<i_{k_n}$ such that

$$F_n\cap\widetilde{S}=\{\tilde{g}_{i_1},\dots,\tilde{g}_{i_{k_n}}\}.$$

Thus,

$$\begin{aligned}
H_\mu(\vee_{g\in F_n\cap\tilde{S}}g^{-1}\alpha)&=H_\mu(\vee_{j=1}^{k_n}\tilde{g}_{i_j}^{-1}\alpha)\\
&=\sum_{r=1}^{k_n}H_\mu(\tilde{g}_{i_j}^{-1}\alpha\mid\vee_{j=1}^{i_r-1}\tilde{g}_{i_j}^{-1}\alpha)\\
&\geq\sum_{r=1}^{k_n}\left(\sum_{B\in\alpha}-\int_X S(1_B)\log\left(S(1_B)\right)d\mu-\frac{1}{2^{i_r}}\right)\\
&\geq|F_n\cap\tilde{S}|\sum_{B\in\alpha}-\int_X S(1_B)\log\left(S(1_B)\right)d\mu-1,
\end{aligned}\tag{15}$$

which implies that

$$h_\mu^{\tilde{S},\mathcal{F}}(G)=\limsup_{n\to\infty}\frac{1}{|F_n\cap\tilde{S}|}H_\mu(\vee_{g\in F_n\cap\tilde{S}}g^{-1}\alpha)\geq\sum_{B\in\alpha}-\int_X S(1_B)\log\left(S(1_B)\right)d\mu.$$

This finishes the proof of Lemma 5.2. □

For further discussion, we need a classical consequence (see [27, Page 94] for example).

Lemma 5.3. *Let $(X, \mathcal{B}_X, \mu)$ be a Borel probability space and $r \geq 1$ be a fixed integer. For each $\epsilon > 0$, there exists $\delta = \delta(\epsilon, r) > 0$ such that if $\alpha = \{A_1, A_2, \ldots, A_r\}$ and $\eta = \{B_1, B_2, \ldots, B_r\}$ are any two finite measurable partitions of $(X, \mathcal{B}_X, \mu)$ with $\sum_{j=1}^r \mu(A_j \Delta B_j) < \delta$, then $H_\mu(\alpha|\eta) + H_\mu(\eta|\alpha) < \epsilon$.*

We need the following result, which is an extension of Lemma 3.3 in [9] from $\mathbb{Z}$-actions to amenable group actions.

Lemma 5.4. *Let $(X, \mathcal{B}_X, \mu, G)$ be a G-MPS, $B \in \mathcal{B}_X$ and F be an infinite subsequence in G. Then the following conditions are equivalent.*

(1) $\overline{\{U_g 1_B : g \in F\}}$ is a compact subset of $L^2(X, \mathcal{B}_X, \mu)$;

(2) for any infinite subsequence $S = \{g_i\}_{i=1}^\infty$ of F,

$$h_\mu^{S,\mathcal{F}}(G, \{B, B^c\}) = 0.$$

Proof. (i) ⇒ (ii). Let $\eta = \{B, B^c\}$. If $\overline{\{U_g 1_B : g \in F\}}$ is a compact subset of $L^2(\mu)$, then for any infinite subsequence $\tilde{S} = \{\tilde{g}_i\}_{i=1}^\infty$ of F, $\overline{\{U_{\tilde{g}_i} 1_B : i \in \mathbb{N}\}}$ is a compact subset of $L^2(\mu)$. So for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that for any $i \geq N$ there exists $n(i) < N$ such that

$$\mu\left(\tilde{g}_i^{-1} B \Delta \tilde{g}_{n(i)}^{-1} B\right) = \|U_{\tilde{g}_i} 1_B - U_{\tilde{g}_{n(i)}} 1_B\|_2 < \delta.$$

For any $\epsilon > 0$, it follows from Lemma 5.3 that for any $i > N$, there exists $n(i) < N$ such that

$$H_\mu\left(\tilde{g}_i^{-1}\eta | \tilde{g}_{n(i)}^{-1}\eta\right) + H_\mu\left(\tilde{g}_{n(i)}^{-1}\eta | \tilde{g}_i^{-1}\eta\right) < \epsilon.$$

Thus for any $i > N$,

$$H_\mu\left(\tilde{g}_i^{-1}\eta | \bigvee_{j=1}^{i-1} \tilde{g}_j^{-1}\eta\right) \leq H_\mu\left(\tilde{g}_i^{-1}\eta | \tilde{g}_{n(i)}^{-1}\eta\right) < \epsilon.$$

By an argument similar to that of (15), we have $h_\mu^{\tilde{S},\mathcal{F}}(G, \eta) \leq \epsilon$. Letting $\epsilon \to 0$, we deduce that $h_\mu^{\tilde{S},\mathcal{F}}(G, \eta) = 0$.

(ii) ⇒ (i). If $\overline{\{U_g 1_B : g \in F\}}$ is not a compact subset of $L^2(X, \mu)$, then there exist $\epsilon > 0$ and an infinite subsequence P of F such that for all $g, h \in P$ with $g \neq h$,

$$\mu\left(g^{-1} B \Delta h^{-1} B\right) = \|U_g 1_B - U_h 1_B\|_2 \geq \epsilon. \tag{16}$$

By the argument at beginning of the section, we obtain an infinite subsequence $S = \{g_i\}_{i=1}^\infty$ of P such that

$$\lim_{i\to\infty} \langle \varphi, U_{g_i}\psi \rangle = \langle \varphi, S(\psi) \rangle \text{ for any } \varphi, \psi \in L^2(X, \mu).$$

It follows from Lemma 5.2 that there exists an infinite subset $\tilde{S} = \{\tilde{g}_i\}_{i=1}^\infty$ of S such that

$$h_\mu^{\tilde{S},\mathcal{F}}(G, \{B, B^c\}) \geq \int_X \left(-S(1_B)\log\left(S(1_B)\right) - S(1_{B^c})\log\left(S(1_{B^c})\right)\right) d\mu.$$

Since $h_\mu^{\tilde{S},\mathcal{F}}(G,\{B,B^c\})=0$, it follows that $-S(1_B)\log S(1_B)=0$ for μ-a.e. $x\in X$. So $S(1_B)$ is an indicator function and $\langle 1_X, S(1_B)\rangle = \|S(1_B)\|_2$. Since μ is an invariant measure, we have $\langle 1_X, 1_B\rangle = \int 1_B d\mu = \lim\limits_{i\to\infty}\int U_{g_i}1_B d\mu = \lim\limits_{i\to\infty}\langle 1_X, U_{g_i}1_B\rangle = \langle 1_X, S(1_B)\rangle$, it follows that $\|S(1_B)\|_2=\|1_B\|_2$. Hence

$$\lim_{i\to\infty}\|U_{g_i}1_B - S(1_B)\|_2 = 0.$$

This implies that for sufficiently large i and j,

$$\mu\left(g_i^{-1}B\Delta g_j^{-1}B\right) < \frac{\epsilon}{2},$$

which contradicts (16). Therefore $\overline{\{U_g 1_B : g\in F\}}$ is a compact subset of $L^2(X,\mu)$. Now we finish the proof of Lemma 5.4. □

Now we characterize the mild mixing via sequence entropy.

Theorem 5.5. *Let $(X,\mathcal{B}_X,\mu,G)$ be a G-MPS. Then the following two conditions are equivalent:*

(1) μ is mildly mixing;

(2) for any $B\in\mathcal{B}_X$ with $0<\mu(B)<1$ and any infinite sequence F in G, there exists an infinite subsequence S of F such that $h_\mu^{S,\mathcal{F}}(G,\{B,B^c\})>0$.

Proof. (1) $\Rightarrow$ (2). Suppose the contrary that there exist $B\in\mathcal{B}_X$ with $0<\mu(B)<1$ and an infinite set F of G such that $h_\mu^{S,\mathcal{F}}(G,\{B,B^c\})=0$ for any infinite subsequence S of F. By Lemma 5.4, we have that $\overline{\{U_g 1_B : g\in F\}}$ is compact in $L^2(X,\mu)$. Thus, there exists an infinite subsequence $\{g_i\}_{i=1}^\infty$ of F such that $\{U_{g_i}1_B\}_{i=1}^\infty$ is a Cauchy sequence in $L^2(X,\mu)$. Using the G-invariance of μ, we deduce that there exists an infinite subsequence $\{\tilde{g}_i\}_{i=1}^\infty$ of G such that

$$\lim_{i\to\infty}\mu(B\Delta\tilde{g}_i^{-1}B) = \lim_{i\to\infty}\|U_{\tilde{g}_i}1_B - 1_B\|_2^2 = 0,$$

which contradicts to the assumption that $(X,\mathcal{B}_X,\mu,G)$ is mildly mixing.

(2) $\Rightarrow$ (1). Fix any $B\in\mathcal{B}_X$ with $0<\mu(B)<1$ and any infinite sequence F of G. If $(X,\mathcal{B}_X,\mu,G)$ is not mildly mixing, then there exists an infinite subsequence $S=\{g_i\}_{i=1}^\infty$ of F such that

$$\lim_{i\to\infty}\|U_{g_i}1_B - 1_B\|_2^2 = \lim_{i\to\infty}\mu(B\Delta g_i^{-1}B) = 0,$$

which implies that $\overline{\{U_{g_i}1_B : i\in\mathbb{N}\}}$ is compact in $L^2(\mu)$. Using Lemma 5.4 again, for any infinite subsequence $\tilde{S}$ of S, one has that $h_\mu^{\tilde{S},\mathcal{F}}(G,\{B,B^c\})=0$, which is a contradiction to (2). □

Now let us proceed to characterize the rigidity of G-MPS.

Theorem 5.6. *Let $(X,\mathcal{B}_X,\mu,G)$ be a G-MPS. Then the following two conditions are equivalent.*

(1) $(X,\mathcal{B}_X,\mu,G)$ is rigid;

(2) there exists an infinite sequence F in G such that for any subsequence S of F, $h_\mu^{S,\mathcal{F}}(G)=0$.

Proof. (1) ⇒ (2). Since $(X, \mathcal{B}_X, \mu, G)$ is rigid, there exists an infinite sequence $F = \{g_n\}_{n=1}^{\infty} \subset G$ such that for any $B \in \mathcal{B}_X$,

$$\lim_{n\to\infty} \mu(B \Delta g_n^{-1} B) = 0. \tag{17}$$

By Lemma 5.4, we only need to prove that for any $B \in \mathcal{B}_X$, $\overline{\{U_g 1_B : g \in F\}}$ is a compact subset of $L^2(X, \mathcal{B}_X, \mu)$. However, this is obtained by the fact that for any $B \in \mathcal{B}_X$,

$$\lim_{n\to\infty} \|1_B - g_n 1_B\|_2^2 = \lim_{n\to\infty} \mu(B \Delta g_n^{-1} B) = 0.$$

(2) ⇒ (1). Using Lemma 5.4 again, we have that $\overline{\{U_g 1_B : g \in F\}}$ is compact in $L^2(X, \mu)$. Thus, there exists an infinite subsequence $\{g_n\}_{n=1}^{\infty}$ of F such that $\{U_{g_n} 1_B\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(X, \mu)$. Using the G-invariance of μ, we deduce that there exists an infinite subsequence $\{\tilde{g}_n\}_{n=1}^{\infty}$ of G such that

$$\lim_{n\to\infty} \mu(B \Delta \tilde{g}_n^{-1} B) = \lim_{n\to\infty} \|U_{\tilde{g}_n} 1_B - 1_B\|_2^2 = 0.$$

The proof is complete. □

Now we have proved Theorem B, which is a combination of Theorems 5.5 and 5.6.

5.2. **Rigid σ-algebra for abelian group actions.** When G is an abelian group, the σ-algebra of rigid sets is G-invariant. Thus, in this subsection, we consider the rigid extension of a G-MPS under the assumption that G is an abelian group. We emphasize that our notion of sequence entropy is defined using any Følner sequence $\mathcal{F}$. Even in the case $G = \mathbb{Z}$, this already generalizes the classical sequence entropy using specific $\mathcal{F} = \{[-n, n] \cap \mathbb{Z} : n \in \mathbb{N}\}$.

Denote by $F(\mathbb{N})$ the family of all finite subsets of $\mathbb{N}$. Given $\alpha, \beta \in F(\mathbb{N})$, we will write $\alpha < \beta$ if every element in α is less than every element in β. The set

$$F^{(1)} := \{\cup_{i\in\beta}\ \alpha_i : \beta \in F(\mathbb{N})\},$$

where $\alpha_1 < \alpha_2 < \ldots \in F(\mathbb{N})$ is called a IP-ring.

We call a sequence in any space Y indexed by $F(\mathbb{N})$ as an $F(\mathbb{N})$-sequence. Let Y be a semigroup. Then an $F(\mathbb{N})$-sequence $\{y_\alpha\}_{\alpha\in F(\mathbb{N})}$ defines an IP-system if $y_\alpha = y_{i_1} \ldots y_{i_k}$ for any $\alpha = \{i_1 < \ldots < i_k\} \in F(\mathbb{N})$. An IP-system can be viewed as a generalized semigroup, as for any $\alpha, \beta \in F(\mathbb{N})$ with $\alpha \cap \beta = \emptyset$ and $\alpha < \beta$, one has $y_{\alpha\cup\beta} = y_\alpha y_\beta$ (see e.g. [4] using this as the definition of IP-systems).

Let $F^{(1)}$ be an IP-ring. Note that any sequence $\{y_\alpha\}_{\alpha\in\ F^{(1)}}$ can be viewed as an $F(\mathbb{N})$-sequence by the following map:

$$\Gamma : F(\mathbb{N}) \to F^{(1)}, \beta \mapsto \cup_{i\in\beta}\alpha_i.$$

Then this map is bijective and satisfies $\Gamma(\alpha \cup \beta) = \Gamma(\alpha) \cup \Gamma(\beta)$. Thus, we can rewrite $\{y_\alpha\}_{\alpha\in F^{(1)}}$ as $\{x_\alpha\}_{\alpha\in F(\mathbb{N})}$.

Now we recall the convergence for $F(\mathbb{N})$-sequences. If $\{x_\alpha\}_{\alpha\in F(\mathbb{N})}$ is an $F(\mathbb{N})$-sequence in a topological space X and $y \in X$, we say x_α converges to y, denoted by

$$\text{IP} - \lim_{\alpha\in F(\mathbb{N})} x_\alpha = y$$

if for any open neighborhood V of y, there is a $\beta(V) \in F(\mathbb{N})$ such that $x_\alpha \in V$ for every $\alpha > \beta(V)$. If X is a compact metric space, there is an IP-ring $F^{(1)}$ such that $\mathrm{IP}-\lim_{\alpha \in F^{(1)}} x_\alpha$ exists.

Theorem 5.7 ([5]). *Let $\{U_\alpha\}_{\alpha \in F(\mathbb{N})}$ be an IP-system of unitary operators on a separable Hilbert space $\mathcal{H}$. Then there is an IP-subsystem $\{U_\alpha\}_{\alpha \in F^{(1)}}$, with $F^{(1)}$ an IP-ring, such that*

$$IP-\lim_{\alpha \in F^{(1)}} U_\alpha = P \textit{ in the weak topology,}$$

where P is the orthogonal projection onto a subspace of $\mathcal{H}$

Let $(X, \mathcal{B}_X, \mu, G)$ be a G-MPS and $\{g_\alpha\}_{\alpha \in F(\mathbb{N})}$ be an IP-system in G. Consider a factor $(Y, \mathcal{B}_Y, \nu, G)$ of $(X, \mathcal{B}_X, \mu, G)$ with the corresponding factor map $\pi : X \to Y$. By Theorem 5.7, there exists an IP-ring $F^{(1)} \subset F(\mathbb{N})$ such that for all $K \in L^2(X \times X, \mathcal{B}_X \otimes \mathcal{B}_X, \mu \times_Y \mu)$,

$$\mathrm{IP}-\lim_{\alpha \in F^{(1)}} g_\alpha K = PK$$

exists in weak topology and P is an orthogonal projection.

Definition 5.8. Let $\{g_\alpha\}$ be an IP-system and $F^{(1)} \subseteq F(\mathbb{N})$ be an IP-ring. A function $f \in L^2(X, \mathcal{B}_X, \mu)$ is called *almost periodic* over $\mathcal{B}_Y$ with respect to g_α along $F^{(1)}$ and one writes $f \in AP(\mathcal{B}_Y, \{g_\alpha\}, F^{(1)})$, if for every $\epsilon > 0$, there exists a set $D \in \mathcal{B}_Y$ with $\nu(D) < \epsilon$ and functions $h_1, \ldots, h_l \in L^2(X, \mathcal{B}_X, \mu)$ such that for every $\delta > 0$ there exists $\alpha_0 \in F^{(1)}$ with the property that whenever $\alpha \in F^{(1)}$ with $\alpha > \alpha_0$ there is a set $E_\alpha \in \mathcal{B}_Y$ with $\nu(E_\alpha) < \delta$ such that for all $y \notin D \cup E_\alpha$,

$$\min_{1 \le j \le l} ||f \circ g_\alpha - h_j||_y < \epsilon.$$

$\overline{AP(\mathcal{B}_Y, \{g_\alpha\}, F^{(1)})}$ is a G-invariant algebra which is invariant under complex conjugacy. Using [31] again, there exists a sub-σ-algebra $\mathcal{K}_F(X|Y) \subset \mathcal{B}_X$, where $F = \{g_\alpha\}_{\alpha \in F^{(1)}}$, such that

$$\overline{AP(\mathcal{B}_Y, \{g_\alpha\}, F^{(1)})} = L^2(X, \mathcal{K}_F(X|Y), \mu).$$

One calls $\mathcal{K}_F(X|Y)$ a rigid algebra over $\mathcal{B}_Y$.

Utilizing similar arguments of Theorem 4.7 and Corollary 4.10 in [2], we have the following results.

Theorem 5.9. *Let $\{g_\alpha\}_{\alpha \in F(\mathbb{N})}$ be an IP-system, $F^{(1)}$ be an IP-ring and $\mathcal{K}_F(X|Y)$ be a rigid algebra over $\mathcal{B}_Y$. Then $f \in L^2(X, \mathcal{K}_F(X|Y), \mu)^\perp$ if and only if*

$$IP-\lim_{\alpha \in F^{(1)}} \int_Y |E(hf \circ g_\alpha | \mathcal{B}_Y)| d\nu = 0$$

for any $h \in L^2(X, \mathcal{B}_X, \mu)$.

Let $\{g_\alpha\}$ be an IP-system and $F^{(1)}$ be an IP-ring. Any subset having the form $\{g_{\beta_i}\}_{i \in \mathbb{N}} \subseteq \{g_\alpha\}_{\alpha \in F^{(1)}}$ with $\beta_1 < \beta_2 < \ldots$ is called an $F(\mathbb{N})$-monotone subset of $\{g_\alpha\}_{\alpha \in F^{(1)}}$.

Theorem 5.10. *Let $f \in L^2(X, \mathcal{B}_X, \mu)$. Then $f \in L^2(X, \mathcal{K}_F(X|Y), \mu)$ if and only if for any $\epsilon > 0$ and any $F(\mathbb{N})$-monotone subset $\{g_{\beta_i}\}_{i\in\mathbb{N}} \subseteq \{g_\alpha\}_{\alpha\in F^{(1)}}$, there exists $M \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n > M$ there is $E_n \in \mathcal{B}_Y$ with $\nu(E_n) < \epsilon$ such that for any $y \notin E_n$,*

$$\inf_{1\leq i\leq n-1} ||g_{\beta_n} - g_{\beta_i}||_y < \epsilon.$$

The following result establishes the relation between relative sequence and rigid σ-algebra.

Theorem 5.11. *Let $(X, \mathcal{B}_X, \mu, G)$ be a G-MPS and $(Y, \mathcal{B}_Y, \nu, G)$ its factor with the factor map π. For any IP-system $\{g_\alpha\}_{\alpha\in F(\mathbb{N})}$ there exists an IP-ring $F^{(1)}$ of $F(\mathbb{N})$ such that for any IP-ring $F^{(2)} \subseteq F^{(1)}$ there exists an $F(\mathbb{N})$-monotone subset $S \subseteq \{g_\alpha\}_{\alpha\in F^{(2)}}$ such that*

$$h_\mu^{S,\mathcal{F}}(G, \xi|\mathcal{B}_Y) = H_\mu(\xi|\mathcal{K}_F(X|Y))$$

for all measurable partitions ξ of X with $H_\mu(\xi|\mathcal{B}_Y) < \infty$ where $F = \{g_\alpha\}_{\alpha\in F^{(1)}}$.

In particular, for any IP-system $\{g_\alpha\}_{\alpha\in F(\mathbb{N})}$ there exists an IP-ring $F^{(1)}$ of $F(\mathbb{N})$ such that

$$\max\{h_\mu^{S,\mathcal{F}}(G, \xi|\mathcal{B}_Y) : S \subseteq F \textit{ is } F(\mathbb{N}) - \textit{monotone}\} = H_\mu(\xi|\mathcal{K}_F(X|Y)).$$

Following ideas in [2], the proof of Theorem 5.11 is obtained by the following four lemmas.

Lemma 5.12. *Let $\{g_\alpha\}_{\alpha\in F(\mathbb{N})}$ be an IP-system. Then there exists an IP-ring $F^{(1)}$ of $F(\mathbb{N})$ such that for any IP-ring $F^{(2)} \subseteq F^{(1)}$ there exists an $F(\mathbb{N})$-monotone subset $S \subseteq \{g_\alpha\}_{\alpha\in F^{(2)}}$ such that*

$$h_\mu^{S,\mathcal{F}}(G, \xi|\mathcal{B}_Y) \geq H_\mu(\xi|\mathcal{K}_F(X|Y))$$

for all measurable partitions ξ of X with $H_\mu(\xi|\mathcal{B}_Y) < \infty$ where $F = \{g_\alpha\}_{\alpha\in F^{(1)}}$.

Proof. Let $F^{(1)}$ be the IP-ring such that IP-$\lim_{\alpha\in F^{(1)}} g_\alpha K = PK$ holds. Consider $\mathcal{K}_F = \mathcal{K}_F(X|Y)$ to be the σ-algebra associated to $F = \{g_\alpha\}_{\alpha\in F^{(1)}}$.

Let $\mathcal{G}_F$ be the family generated by $\{\{g_\alpha\}_{\alpha\in F^{(2)}} : F^{(2)}$ is an IP-ring of $F^{(1)}\}$. By Hindman's theorem $\mathcal{G}_F^*$ is a filter.

Claim: For any measurable finite partition ξ and η of X and $\epsilon > 0$, there exists a sequence $S \in \mathcal{G}_F^*$ such that for any $g \in S$

$$\int_Y H_y(g^{-1}\xi|\eta)d\nu \geq H_\mu(\xi|\mathcal{K}_F) - \epsilon.$$

Proof of Claim: Note that

(18)

$$\mathcal{G}_F^* = \{S \subseteq G : \text{ there exists } \alpha_0 \in F^{(1)} \text{ such that } g_\alpha \in S \text{ for any } \alpha \in F^{(1)} \text{ with } \alpha > \alpha_0\}.$$

For any $A, B \in \mathcal{B}_X$, since $1_A - E(1_A|\mathcal{K}_F) \in L^2(X, \mathcal{K}_F, \mu)^\perp$, it follows from Theorem 5.9 that

$$\text{IP} - \lim_{\alpha\in F^{(1)}} \int_Y |\int_X (1_A - E(1_A|\mathcal{K}_F)) \circ g_\alpha 1_B d\mu_y| d\nu = 0,$$

which together with (18), implies that

$$\mathcal{G}_F^* - \lim \int_Y |\int_X (1_A - E(1_A|\mathcal{K}_F)) \circ g_n 1_B d\mu_y| d\nu = 0.$$

Let $\xi = \{A_1, \ldots, A_k\}$ and $\eta = \{B_1, \ldots, B_l\}$. Since $\mathcal{G}_F^*$ is a filter, for any fixed $\delta_1, \delta_2 > 0$, there exists $S \in \mathcal{G}_F^*$ such that for any $g \in S$ there is a set $E_g \in \mathcal{B}_Y$ with $\nu(E_g) > 1 - \delta_1$ satisfying

$$|\mu(g^{-1}A_i \cap B_j) - \int_X E(1_{A_i}|\mathcal{K}_F) \circ g 1_{B_j} d\mu_y| < \delta_2$$

for all $1 \leq i \leq k, 1 \leq j \leq l$ and $y \in E_g$. The proof of Claim is completed by a similar argument to that of Lemma 4.3. □

For any IP-ring $F^{(2)} \subseteq F^{(1)}$, $S' = \{g_\alpha\}_{\alpha \in F^{(2)}}$ is an IP-ring and $S' \in \mathcal{G}_F$. Thus one can complete the proof in the same way as the proof of Lemma 4.3. □

The following result is classical (see e.g. [27, Page 94]).

Lemma 5.13. *Let $r \geq 1$ be a fixed integer. For each $\epsilon > 0$, there exists $\delta > 0$ such that if $\xi = \{A_1, \ldots, A_r\}$ and $\eta = \{C_1, \ldots, C_r\}$ are two measurable partitions of X with $\sum_{i=1}^r \mu(A_i \triangle C_i) < \delta$ then the Rokhlin metric satisfies*

$$\rho_\mu(\xi, \eta) = H_\mu(\xi|\eta) + H_\mu(\eta|\xi) < \epsilon.$$

Lemma 5.14. *Let $\{g_\alpha\}_{\alpha \in F(\mathbb{N})}$ be an IP-system and $B \in \mathcal{B}_X$. Then there exists an IP-ring $F^{(1)}$ of $F(\mathbb{N})$ such that $B \in \mathcal{K}_F(X|Y)$ if and only if*

$$h_\mu^{S,\mathcal{F}}(G, \{B, B^c\}|\mathcal{B}_Y) = 0$$

for all $F(\mathbb{N})$-monotone subsets $S \subseteq F$ where $F = \{g_\alpha\}_{\alpha \in F^{(1)}}$.

Proof. Let $F^{(1)}$ be such that $\mathrm{IP}-\lim_{\alpha \in F^{(1)}} g_\alpha K = PK$ holds and assume $B \in \mathcal{K}_F(X|Y)$. Let $A = \{g_{\beta_i}\}_{i \in \mathbb{N}}$ be an $F(\mathbb{N})$-monotone subset of $F = \{g_\alpha\}_{\alpha \in F^{(1)}}$ and $\xi = \{B, B^c\}$.

Observe that $||g_1 1_B - g_2 1_B||_y^2 = \mu_y(g_1^{-1}B \triangle g_2^{-1}B)$ for any $g_1, g_2 \in G$. Since $1_B \in L^2(X, \mathcal{K}_F(X|Y), \mu)$, there is $M \in \mathbb{N}$ such that for any $n > M$, there exists $E_n \in \mathcal{B}_Y$ with $\nu(E_n) < \delta$ satisfying, for any $y \notin E_n$,

$$\inf_{1 \leq j \leq n-1} ||1_B \circ g_{\beta_n} - 1_B \circ g_{\beta_j}||_y < \delta.$$

Moreover, given any $\epsilon > 0$, the above δ is chosen according to Lemma 5.13.

Hence for any $n > M$, there is $1 < j(n) < n - 1$ such that $\rho_y(g_{\beta_n}^{-1}\xi, g_{\beta_{j(n)}}^{-1}\xi) < \epsilon$. So

$$H_y(g_{\beta_n}^{-1}\xi | \bigvee_{i=1}^{n-1} g_{\beta_i}^{-1}\xi) \leq H_y(g_{\beta_n}^{-1}\xi | g_{\beta_{j(n)}}^{-1}\xi) \leq \epsilon.$$

The rest of proof is similar to that of Lemma 4.4. □

Lemma 5.15. *Let $\{g_\alpha\}_{\alpha \in F(\mathbb{N})}$ be an IP-system. Then there exists an IP-ring $F^{(1)}$ of $F(\mathbb{N})$ such that for any finite measurable partition ξ of X and $F(\mathbb{N})$-monotone subset $S \subseteq F$*

$$h_\mu^{S,\mathcal{F}}(G, \xi|\mathcal{B}_Y) \leq H_\mu(\xi|\mathcal{K}_F(X|Y)),$$

where $F = \{g_\alpha\}_{\alpha \in F^{(1)}}$.

Proof. Let $F^{(1)}$ be such that $\mathrm{IP}-\lim_{\alpha\in F^{(1)}} g_\alpha K = PK$ holds. The proof is analogous to that of Lemma 4.5, by replacing $\mathcal{K}(X|Y)$ by $\mathcal{K}_F(X|Y)$. □

Now we are able to characterize the rigid and mild mixing extension via relative sequence entropy. Let us begin with some definitions. Referring to [5, 2], we have the following definition.

Definition 5.16. Let $(X,\mathcal{B}_X,\mu,G)$ be a G-MPS and $(Y,\mathcal{B}_Y,\nu,G)$ be its factor. Then

(1) $(X,\mathcal{B}_X,\mu,G)$ is called a *rigid extension* of $(Y,\mathcal{B}_Y,\nu,G)$ if there is an IP-system F such that $\mathcal{K}_F(X|Y)=\mathcal{B}_X$;

(2) $(X,\mathcal{B}_X,\mu,G)$ is called a *mild mixing extension* of $(Y,\mathcal{B}_Y,\nu,G)$ if there exists an IP-system F such that $\mathcal{K}_F(X|Y)=\mathcal{B}_Y$.

Finally, Theorem B', restated in the following, is a corollary of Theorem 5.11.

Theorem 5.17. *Let $(X,\mathcal{B}_X,\mu,G)$ be a G-MPS and $(Y,\mathcal{B}_Y,\nu,G)$ one of its factor where G is an abelian group. Then*

(1) $(X,\mathcal{B}_X,\mu,G)$ is a rigid extension of $(Y,\mathcal{B}_Y,\nu,G)$ if and only if there exists an IP-system $F=\{g_\alpha\}_{\alpha\in F(\mathbb{N})}$ such that $h_\mu^{S,\mathcal{F}}(G|\mathcal{B}_Y)=0$ for any $F(\mathbb{N})$-monotone subset $S\subseteq F$;

(2) $(X,\mathcal{B}_X,\mu,G)$ is a mild mixing extension of $(Y,\mathcal{B}_Y,\nu,G)$ if and only if any IP-system $F=\{g_\alpha\}_{\alpha\in F(\mathbb{N})}$, there exists an $F(\mathbb{N})$-monotone subset $S\subseteq F$ such that $h_\mu^{S,\mathcal{F}}(G,\xi|\mathcal{B}_Y)=H_\mu(\xi|\mathcal{B}_Y)$ for any measurable partition ξ of X with $H_\mu(\xi|\mathcal{B}_Y)<\infty$.

6. RELATIVE TOPOLOGICAL SEQUENCE ENTROPY AND WEAKLY MIXING SET

In this section, we aim to establish the characterization of weakly mixing sets via relative topological sequence entropy for amenable group actions. To be specific, we extend the results by Oprocha and Zhang that the existence of weakly mixing set implies infinite sequence entropy [22] from $\mathbb{Z}$-actions to amenable group actions.

Let us begin with some notations. Given two covers $\mathcal{C}=\{C_1,\dots,C_k\}$ and $\mathcal{D}=\{D_1,\dots,D_l\}$, denote

$$\mathcal{C}\vee\mathcal{D}:=\{C_i\cap D_j: i=1,\dots,k, \text{ and } j=1,\dots,l, \text{ where } C_i\cap D_j\neq\emptyset\}.$$

Now consider a finite cover $\mathcal{U}$ of X, a non-empty set $K\subseteq X$, and a sequence $S\subseteq G$. Define $N(\mathcal{U}|K)$ as the smallest number of sets from $\mathcal{U}$ needed to cover K. The *topological sequence entropy of K with respect to the sequence S and the Følner sequence $\mathcal{F}$* is given by:

$$h_{top}^{S,\mathcal{F}}(G,K)=\sup_{\mathcal{U}} h_{top}^{S,\mathcal{F}}(G,\mathcal{U},K),$$

where the supremum is taken over all finite open covers $\mathcal{U}$ of K and

$$h_{top}^{S,\mathcal{F}}(G,\mathcal{U},K):=\limsup_{n\to\infty}\frac{1}{|F_n\cap S|}\log N(\bigvee_{g\in F_n\cap S} g^{-1}\mathcal{U}|K).$$

Let (Y,G) be a factor of (X,G) with the factor map π and $\mathcal{U}$ be a finite open cover of X. For any $y\in Y$, $\pi^{-1}(y)$ is a subset of X, and denote

$$N(\mathcal{U}|y):=N(\mathcal{U}|\pi^{-1}y).$$

Then we can define the *relative topological sequence entropy of S along Følner sequence $\mathcal{F}$* by

$$h_{top}^{S,\mathcal{F}}(G,\mathcal{U}|\pi) = \sup_{y\in Y} h_{top}^{S,\mathcal{F}}(G,\mathcal{U}|y),$$

where

$$h_{top}^{S,\mathcal{F}}(G,\mathcal{U}|y) = \limsup_{n\to\infty} \frac{1}{|F_n\cap S|}\log N(\bigvee_{g\in F_n\cap S} g^{-1}\mathcal{U}|y).$$

Then the *relative topological sequence entropy of* (X,G) is defined as

$$h_{top}^{S,\mathcal{F}}(G,X|\pi) = \sup_{\mathcal{U}} h_{top}^{S,\mathcal{F}}(G,\mathcal{U}|\pi),$$

where the supremum is taken over all finite open covers $\mathcal{U}$ of X. In the case of π being trivial, i.e. Y is a singleton, then the relative topological sequence entropy is the topological sequence entropy.

Definition 6.1. Let (X,G) be a G-system and $\emptyset \neq A \subseteq X$, $n>1$. We say A is

(1) *transitive* if for each pair of open subsets (U,V) of X intersecting A, there exists $g\in G$ with $g^{-1}(V\cap A)\cap U\neq\emptyset$;
(2) *weakly mixing of order* n if A^n is transitive set of (X^n,G). Here, X^n denotes the n-fold product space of X, and $g\in G$ acts naturally on X^n by: $(x_1,\dots,x_n)\mapsto(gx_1,\dots,gx_n)$;
(3) *weakly mixing* if A is weakly mixing of each order $n>1$.

Here we denote the recurrence time set of $A,B\subseteq X$ by

$$\tau(A,B) := \{g\in G : A\cap g^{-1}B\neq\emptyset\}.$$

We restate and prove Theorem C as follows.

Theorem 6.2. *Let (X,G) be a G-system. If A is a weakly mixing set, then the sequence entropy of A is infinite, i.e., for any $m\in\mathbb{N}$, there is a sequence S of G such that $h_{top}^{S,\mathcal{F}}(G,A)\geq\log m$.*

Proof. Let $m\geq 2$. Choose open sets $V_1,U_1,\dots,V_m,U_m$ intersecting with A such that $\overline{V_i}\subseteq U_i$ and $U_i\cap U_j=\emptyset$ whenever $1\leq i<j\leq m$.

Claim: There exist a sequence $S=\{g_n\}_{n=1}^\infty$ of distinct elements of G and an increasing sequence $\{m_n\}_{n=1}^\infty$ of positive integers such that for each $n\geq 1$, $g_n\in F_{m_n+1}\backslash F_{m_n}$ and for any $s\in\{1,2,\dots,m\}^n$

$$\bigcap_{i=1}^{n} g_i^{-1}V_{s(i)}\cap A\neq\emptyset.$$

Proof of the claim. It is obvious that the claim holds for $n=1$ by choosing $g_1=e_G$. Assume that there exist $g_1,\dots,g_k\in G$ and $m_1<m_2<\dots<m_k$ such that for each $i\in\{1,2,\dots,k\}$, $g_i\in F_{m_i+1}\backslash F_{m_i}$ and

$$\bigcap_{i=1}^{k} g_i^{-1}V_{s(i)}\cap A\neq\emptyset \text{ for all } s\in\{1,2,\dots,m\}^k.$$

Take $m_{k+1} \in \mathbb{N}$ large enough, so that $g_1, \ldots, g_k \in F_{m_k+1}$. Since A is weakly mixing of any order n, the set

$$\bigcap_{s\in\{1,2,\ldots,m\}^k} \bigcap_{j=1}^{m} \tau(\bigcap_{i=1}^{k} g_i^{-1} V_{s(i)} \cap A, V_j),$$

is an infinite set (see [28]), and hence there exists $g_{k+1} \notin F_{m_{k+1}}$ with $g_{k+1} \neq g_i$ for each $i = 1, 2, \ldots, k$ such that

$$g_{k+1} \in \bigcap_{s\in\{1,2,\ldots,m\}^k} \bigcap_{j=1}^{m} \tau(\bigcap_{i=1}^{k} g_i^{-1} V_{s(i)} \cap A, V_j).$$

Through this iterative process, we obtain the sequence $S = \{g_n\}_{n=1}^{\infty}$ and an increasing sequence $\{m_n\}_{n=1}^{\infty}$ of positive integers satisfying the claim. □

Define $W = X \setminus (\overline{V_1} \cup \cdots \cup \overline{V_m})$. Now, consider the collection $\mathcal{U}$ defined as $\mathcal{U} = \{W \cup U_1, \ldots, W \cup U_m\}$, which is an open cover of X. Take the infinite subset S as in the claim, and for any $n \in \mathbb{N}$ and $s \in \{1, \ldots, m\}^n$, choose $x_s \in \bigcap_{i=1}^{n} g_i^{-1} V_{s(i)} \cap A$. Denote $A_n = \{x_s : s \in \{1, \ldots, m\}^n\} \subseteq A$. Then $|A_n| = m^n$ and each element of $\bigvee_{i=1}^{n} g_i^{-1}\mathcal{U}$ contains exactly one point from A_n. It implies that $N(\bigvee_{i=1}^{n} g_i^{-1}\mathcal{U}|A) = m^n$. Therefore,

$$\begin{aligned} h_{top}^{S,\mathcal{F}}(G, A) \geq h_{top}^{S,\mathcal{F}}(G, A, \mathcal{U}) &= \limsup_{n\to\infty} \frac{1}{|F_n \cap S|} \log N(\bigvee_{g\in F_n\cap S} g^{-1}\mathcal{U}|A) \\ &\geq \limsup_{n\to\infty} \frac{1}{|F_{m_{n+1}} \cap S|} \log N(\bigvee_{g\in F_{m_{n+1}}\cap S} g^{-1}\mathcal{U}|A) \\ &\geq \log m. \end{aligned}$$

The proof is completed. □

Using this theorem, it is easy to obtain the following, which implies Corollary C.1.

Corollary 6.3. *Let (X, G) be a G-system and (Y, G) be its factor with factor map π. Then for any $y \in Y$, if $\pi^{-1}(y)$ is weakly mixing, then for any $m \in \mathbb{N}$, there is a sequence S of G such that $h_{top}^{\mathcal{F},S}(G, y) > \log m$.*

7. Relative sequence entropy pair

At the end, we consider relative sequence entropy from the local viewpoint. Let $(X, \mathcal{B}_X, \mu, G)$ be a G-MPS and $(Y, \mathcal{B}_Y, \nu, G)$ be its factor with factor map π.

For any subset A of X, denote by A^o the interior of the set A. A pair $(x, y) \in X \times X$ with $x \neq y$ is called a *relative sequence entropy pair* if for any Borel partition $\xi = \{A, A^c\}$ of X with $x \in A^o$ and $y \in (A^c)^o$, there exists an infinite sequence S such that $h_\mu^{S,\mathcal{F}}(G, \xi|\mathcal{B}_Y) > 0$, i.e. $H_\mu(\xi|\mathcal{K}(X|Y)) > 0$ by Theorem A. We denote the set of relative sequence entropy pairs by $SE_{(Y,\mu)}$.

Denote $(X, \mathcal{K}(X|Y), \mu)$ by (Z, m) for convenience. Let $\mu = \int_Z \mu_z dm(z)$ be the disintegration of μ over (Z, m). Set

$$\lambda_\mu = \int_Z (\mu_z \times \mu_z) dm(z),$$

the relatively independent joining of μ over m. Let Λ_μ be the topological support of λ_μ.

Theorem 7.1. *For a measure $\mu \in M(X, G)$, $SE_{(Y,\mu)} = \Lambda_\mu \backslash \Delta$ where Δ is the diagonal of $X \times X$.*

Proof. For convenience, we write $\mathcal{K}(X|Y)$ as $\mathcal{K}$ in the following proof. Suppose $(x, y) \notin SE_{(Y,\mu)} \cup \Delta$. Then there exists a Borel partition $\xi = \{A, A^c\}$ of X such that $x \in A^o$, $y \in (A^c)^o$ and $H_\mu(\xi|\mathcal{K}) = 0$. This implies that A is $\mathcal{K}$-measurable. Hence

$$\lambda_\mu(A \times A^c) = \int \mu_z(A)\mu_z(A^c) dm(z) = 0.$$

Thus $(x, y) \notin \Lambda_\mu$.

Conversely, suppose $(x, y) \notin \Lambda_\mu \cup \Delta$. Then there exist disjoint open neighborhoods A and B of x and y respectively with

$$0 = \lambda_\mu(A \times B) = \int \mu_z(A)\mu_z(B) dm(z) = \int E(1_A|\mathcal{K})E(1_B|\mathcal{K}) d\mu(x). \tag{19}$$

Let $D = \{x \in X : E(1_A|\mathcal{K})(x) = 0\} \in \mathcal{K}$. Then by (19), $E(1_B|\mathcal{K})(x) = 0$, for μ-a.e. $x \in D^c$. Thus, $\mu(B \cap D^c) = E(E(1_B|\mathcal{K})1_{D^c}) = 0$, which implies that $B \subset D$. Similarly, $A \subset D^c$. Then we have that D is in $\mathcal{K}$ such that $H_\mu(\{D^o, (D^c)^o\}|\mathcal{K}) = 0$. Thus $(x, y) \notin SE_{(Y,\mu)}$. Hence the proof is complete. □

A pair (x_1, x_2) is referred to as a *topological sequence entropy pair relative to* Y if for any closed, disjoint neighborhoods A and B of x_1 and x_2, respectively, there exists an infinite subset $S \subset G$ such that $h_{top}^{S,\mathcal{F}}(G, \{A^c, B^c\}|\pi) > 0$. Denote the set of all sequence entropy pairs by $SE_Y(X, G)$. We can use topological relative sequence entropy pairs to describe null G-system relative to Y .

Theorem 7.2. *Let (X, G) be a G-system and (Y, G) be its factor. Then (X, G) is null relative to Y if and only if $SE_Y(X, G) = \emptyset$.*

Proof. It is clear that if (X, G) is null relative to Y then $SE_Y(X, G) = \emptyset$.

Conversely, we have the following claim.

Claim: If there is an open cover $\mathcal{U} = \{U, V\}$ of X such that $h_{top}^{S,\mathcal{F}}(G, \mathcal{U}|\pi) > 0$ for some infinite sequence $S \subset G$, then there are points $x \in U^c$ and $y \in V^c$ such that (x, y) is a sequence entropy pair relative to Y.

Proof of the claim. Let $\mathcal{U} = \{U, V\}$ be such that $h_{top}^{S,\mathcal{F}}(G, \mathcal{U}|\pi) > 0$. By an argument similar to that of [19, Lemma 5.3], there is a sequence of open covers $\mathcal{U}_n = \{U_n, V_n\}$ such that for each $n \in \mathbb{N}$

(a) $\mathcal{U}_n \succeq \mathcal{U}_{n+1}$;
(b) the diameters of U_n^c and V_n^c converge to 0;

(c) $h^{S,\mathcal{F}}_{top}(G,\mathcal{U}_n|\pi) > 0$.

By the compactness of X and items (a) (b), there exist $x, y \in X$ such that

$$\cap_{n=1}^{\infty} U_n^c = \{x\} \text{ and } \cap_{n=1}^{\infty} U_n^c = \{y\}.$$

For any disjoint closed neighborhoods $x \in A$ and $y \in B$, there is an $n \in \mathbb{N}$ such that $U_n^c \subseteq A$ and $V_n^c \subseteq B$, which together with (c) implies that and $h^{S,\mathcal{F}}_{top}(G, A^c, B^c|\pi) \geq h^{S,\mathcal{F}}_{top}(G,\mathcal{U}_n|\pi) > 0$. Therefore, (x, y) is a sequence entropy pair relative to Y. The proof of claim is completed. □

Thus, if (X, G) is not null, that is, there is an open cover $\mathcal{U}$ of X such that

$$h^{S,\mathcal{F}}_{top}(G,\mathcal{U}|\pi) > 0,$$

then by the claim we have $SE_Y(X, G) \neq \emptyset$. The proof is completed. □

References

[1] François Blanchard and Wen Huang. Entropy sets, weakly mixing sets and entropy capacity. *Discrete Contin. Dyn. Syst.*, 20(2):275–311, 2008.

[2] Alvaro Coronel, Alejandro Maass, and Song Shao. Sequence entropy and rigid σ-algebras. *Studia Math.*, 194(3):207–230, 2009.

[3] Hillel Furstenberg. *Recurrence in ergodic theory and combinatorial number theory*. Princeton University Press, Princeton, N.J., 1981. M. B. Porter Lectures.

[4] Hillel Furstenberg. IP-systems and recurrence in ergodic theory: an update. *Israel J. Math.*, 251(2):423–441, 2022.

[5] Hillel Furstenberg and Yitzhak. Katznelson. An ergodic Szemerédi theorem for IP-systems and combinatorial theory. *J. Analyse Math.*, 45:117–168, 1985.

[6] Eli Glasner. *Ergodic theory via joinings*, volume 101 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[7] Tim Goodman. Topological sequence entropy. *Proc. London Math. Soc. (3)*, 29:331–350, 1974.

[8] Wen Huang, Alejandro Maass, and Xiangdong Ye. Sequence entropy pairs and complexity pairs for a measure. *Ann. Inst. Fourier (Grenoble)*, 54(4):1005–1028, 2004.

[9] Wen Huang, Song Shao, and Xiangdong Ye. Mixing via sequence entropy. In *Algebraic and topological dynamics*, volume 385 of *Contemp. Math.*, pages 101–122. Amer. Math. Soc., Providence, RI, 2005.

[10] Wen Huang and Xiangdong Ye. Combinatorial lemmas and applications to dynamics. *Adv. Math.*, 220(6):1689–1716, 2009.

[11] Paul Hulse. Sequence entropy and subsequence generators. *J. London Math. Soc. (2)*, 26(3):441–450, 1982.

[12] Paul Hulse. Sequence entropy relative to an invariant σ-algebra. *J. London Math. Soc. (2)*, 33(1):59–72, 1986.

[13] David Kerr and Hanfeng Li. Combinatorial independence in measurable dynamics. *J. Funct. Anal.*, 256(5):1341–1386, 2009.

[14] Bernard Osgood Koopman and John von Neumann. Dynamical systems of continuous spectra. *Proceedings of the National Academy of Sciences*, 18(3):255–263, 1932.

[15] Aleksandr Kušnirenko. Metric invariants of entropy type. *Uspehi Mat. Nauk*, 22(5 (137)):57–65, 1967.

[16] Simin Li. Topological sequence entropy and topologically weak mixing. *Bull. Austral. Math. Soc.*, 66(2):207–211, 2002.

[17] Elon Lindenstrauss. Pointwise theorems for amenable groups. *Invent. Math.*, 146(2):259–295, 2001.

[18] Chunlin Liu, Xiangtong Wang, and Leiye Xu. Sequence entropy and IT-tuples for minimal group actions. *Adv. Math.*, 467:Paper No. 110183, 2025.

[19] Chunlin Liu and Kesong Yan. Sequence entropy for amenable group actions. *Physica Scripta*, 98(12):125268, dec 2023.
[20] Adam Matthew Lott. *Relative Slow Entropy and Applications*. ProQuest LLC, Ann Arbor, MI, 2023. Thesis (Ph.D.)–University of California, Los Angeles.
[21] Piotr Oprocha and Guohua Zhang. On local aspects of topological weak mixing in dimension one and beyond. *Studia Math.*, 202(3):261–288, 2011.
[22] Piotr Oprocha and Guohua Zhang. On local aspects of topological weak mixing, sequence entropy and chaos. *Ergodic Theory Dynam. Systems*, 34(5):1615–1639, 2014.
[23] William Parry. *Topics in ergodic theory*, volume 75 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2004. Reprint of the 1981 original.
[24] Alan Saleski. Sequence entropy and mixing. *J. Math. Anal. Appl.*, 60(1):58–66, 1977.
[25] Iddo Samet. Rigid actions of amenable groups. *Israel J. Math.*, 173:61–90, 2009.
[26] Klaus Schmidt and Peter Walters. Mildly mixing actions of locally compact groups. *Proc. London Math. Soc. (3)*, 45(3):506–518, 1982.
[27] Peter Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.
[28] Zhaolong Wang and Guohua Zhang. Chaotic behavior of group actions. In *Dynamics and numbers*, volume 669 of *Contemp. Math.*, pages 299–315. Amer. Math. Soc., Providence, RI, 2016.
[29] Qing Zhang. Sequence entropy and mild mixing. *Canad. J. Math.*, 44(1):215–224, 1992.
[30] Qing Zhang. Conditional sequence entropy and mild mixing extensions. *Canad. J. Math.*, 45(2):429–448, 1993.
[31] Robert Zimmer. Extensions of ergodic group actions. *Illinois J. Math.*, 20(3):373–409, 1976.

School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, P.R. China
Email address: chunlinliu@mail.ustc.edu.cn

School of Mathematical Sciences, Xiamen University, Xiamen, 361005, P.R. China
Email address: clwang@stu.xmu.edu.cn

School of Mathematical Sciences, Xiamen University, Xiamen, 361005, P.R. China
Email address: wuweisheng@xmu.edu.cn