A NOTE ON MULTIVARIATE DIAM-MEAN EQUICONTINUITY AND FREQUENT STABILITY

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Abstract. Let (X, G) be a topological dynamical system, given by the action of a countable discrete infinite group on a compact metric space X. We prove that every minimal system (X, G) is either diam-mean m-equicontinuous or diam-mean m-sensitive. We further introduce the notion of strong m-spreading and show that every minimal system (X, G) is either frequently m-stable or strongly m-spreading. Further, when G is abelian (or, more generally, (virtually) nilpotent), then the following statements are equivalent:

(1) (X, G) is a regular m-to-one extension of its maximal equicontinuous factor;
(2) (X, G) is diam-mean $(m+1)$-equicontinuous, and not diam-mean m-equicontinuous;
(3) (X, G) is not diam-mean $(m+1)$-sensitive, but diam-mean m-sensitive;
(4) (X, G) has an essential weakly mean sensitive m-tuple but no essential weakly mean sensitive $(m+1)$-tuple.

This provides a *'local'* characterisation of m-regularity and mean m-sensitivity via weakly mean sensitive tuples. The same result holds when G is amenable and (X, G) satisfies the local Bronstein condition.

1. Introduction

Notions of sensitivity and – contrastingly – corresponding notions of equicontinuity have been studied for a long time in topological dynamics. A basic question is whether points that start close to each other remain close under the action, or whether small perturbations can eventually lead to large deviations. A classical result in this direction is the Auslander–Yorke dichotomy [4], which states that a minimal topological dynamical system is either equicontinuous or sensitive. This means that, in the minimal setting, any system that is not fully regular – in the sense of equicontinuity – already exhibits a definite form of instability.

Equicontinuous systems form one of the best understood classes of dynamical systems. They are equivalently characterised by the fact that they admit a topologically equivalent metric that is invariant under the action. In the case of abelian acting groups, minimal equicontinuous systems can also be described as rotations on compact groups [1, Chapter 3, Theorem 6]. By contrast, sensitive dependence on initial conditions is a rather weak notion: it only requires that arbitrarily close initial points separate by some fixed positive distance at least once along their orbits. It does not exclude the possibility that these points remain close for most times on average.

This led to the study of mean versions of equicontinuity and sensitivity. These notions were introduced and investigated in [12, 24]. For minimal systems, mean equicontinuity has a clear structural meaning: it is equivalent to the factor map onto the maximal equicontinuous factor (MEF) being a measure-theoretic isomorphism. In this way, a weakened form of equicontinuity corresponds to the structure of the fibres over the maximal equicontinuous factor. Mean equicontinuous systems include several well-known examples, such as regular Toeplitz flows [11, 32], regular model sets [6, 30], and symbolic systems such as Sturmian subshifts or primitive substitutions with discrete spectrum [5].

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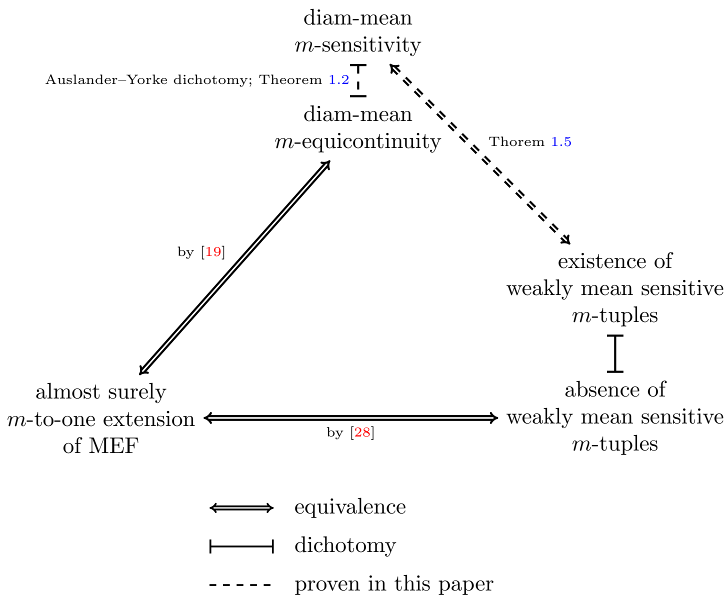

FIGURE 1. Diagram of our results

They are also relevant in the study of aperiodic order, where they appear as mathematical models with strong long-range order, for example in the study of quasicrystals [5]. An Auslander–Yorke type dichotomy for mean equicontinuity was established in [24].

Several weaker forms of equicontinuity and stronger forms of sensitivity have been studied since then. Among them are diam-mean equicontinuity and frequent stability. For minimal systems, these notions also admit structural descriptions in terms of the fibres over the maximal equicontinuous factor. More precisely, a minimal extension of an equicontinuous system is regular, that is, almost surely one-to-one, if and only if it is diam-mean equicontinuous, and it is almost one-to-one, that is, one-to-one on a residual set, if and only if it is frequently stable [15, 20]. These results show again that forms of equicontinuity correspond to restrictions on the fibre structure over the MEF.

More recently, multivariate forms of mean equicontinuity and sensitivity have also been studied [21, 25, 26, 29, 31]. These notions do not only consider pairs of nearby points, but the joint behaviour of several points at the same time. In the minimal setting, multivariate mean equicontinuity is characteristic of measure-theoretically finite-to-one extensions of equicontinuous systems [10]. This class again contains many standard examples, including the Thue–Morse substitution and related systems [5], certain irregular Toeplitz flows constructed by Williams [32], Iwanik and Lacroix [23], and certain irregular model sets [14]. An Auslander–Yorke dichotomy for multivariate mean equicontinuity was proven in [10].

The purpose of this paper is to continue this line of research for the multivariate analogues of diam-mean equicontinuity and frequent stability introduced in [19]. Our first main result establishes Auslander–Yorke type dichotomies for these notions. Our second main result gives a local characterisation of multivariate mean sensitivity in terms of weakly mean sensitive tuples. These tuples were studied in [28], where their absence was used to characterise almost-surely finite-to-one extensions of equicontinuous systems. Combined with the results of [19], this already implies a close relation to multivariate diam-mean equicontinuity. Our argument provides a direct proof of this relation and thus gives an additional point of view on the connection. See Figure 1 for a summary of the relations between the different notions.

1.1. **Main results.** In order to state our main results, we need to provide some definitions. We call a pair (X, G) a *topological dynamical system* (tds) if X is a compact metric space and G is a countably infinite discrete amenable group that acts on X by homeomorphisms. With slight abuse of notation we write $g : X \to X,\ x \mapsto gx$ for $g \in G$. A tds is called *transitive* if there exists a point with dense orbit and *minimal* if all orbits are dense. A point $x \in X$ is called an *almost periodic point* (or a *minimal point*) if the closure of its orbit is a minimal system.

It is well-known that any tds has a *maximal equicontinuous factor (MEF)* [1, 11], which we denote by (X_{eq}, G). The corresponding factor map will be denoted by $\pi_{\mathrm{eq}} : X \to X_{\mathrm{eq}}$. When (X, G) is transitive, its MEF is strictly ergodic with a unique invariant Borel probability measure μ_{eq}. We call (X, G) *m-regular* if $\mu_{\mathrm{eq}}\left(\left\{y \in X_{\mathrm{eq}} \mid \left|\pi_{\mathrm{eq}}^{-1}(\{y\})\right| = m\right\}\right) = 1$ and *almost m:1* if $\left\{y \in X_{\mathrm{eq}} \mid \left|\pi_{\mathrm{eq}}^{-1}(\{y\})\right| = m\right\}$ is residual.

The dynamical properties corresponding to such m:1-extensions are naturally formulated in terms of $(m+1)$-tuples of points. Accordingly, the terminology in the literature uses the index $m+1$. For instance, the classical notion of diam-mean equicontinuity is referred to as diam-mean 2-equicontinuity; there is no notion of diam-mean 1-equicontinuity. While this index shift is natural from this perspective, it may also cause confusion. To avoid ambiguity we distinguish between 1-based and 2-based indexing by introducing the notation

$$\mathbb{N}_1 := \{n \in \mathbb{N} : n \geq 1\},$$
$$\mathbb{N}_2 := \{n \in \mathbb{N} : n \geq 2\}.$$

Given $m \in \mathbb{N}_2$, we define the *m-separation* of m points $x_1, \dots, x_m \in X$ as

$$D_m(x_1, \dots x_m) \ = \ \min_{1 \leq i < j \leq m} d(x_i, x_j) \ ,$$

where d denotes the metric on X. Given a subset $A \subseteq X$, the m-diameter of A is given by

$$\mathrm{diam}_m(A) \ = \ \sup \left\{ D_m(x_1, \dots, x_m) \mid (x_1, \dots, x_m) \in A^m \right\} \ .$$

A tds (X, G) is said to be *diam-mean m-equicontinuous*[1] if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all sets $U \subseteq X$ with $\mathrm{diam}(U) := \mathrm{diam}_2(U) < \delta$ we have

$$\sup_{\mathcal{F}} \limsup_{n \to \infty} \frac{1}{|\mathcal{F}_n|} \sum_{g \in \mathcal{F}_n} \mathrm{diam}_m(gU) < \ \varepsilon \ . \tag{1.1}$$

Here and in all of the following $\sup_{\mathcal{F}}$ is the supremum taken over all Følner sequences $\mathcal{F}$ of G. Furthermore, we understand $\mathcal{F}_n$ to be the n-th element of $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{N}}$. The significance of this dynamical property of diam-mean m-equicontinuity lies in the fact that — at least in the case of minimal abelian group actions — a tds is m-regular if and only if it is diam-mean $(m+1)$-equicontinuous, but not diam-mean m-equicontinuous [19].

Remark 1.1. We briefly comment on a minor notational difference with [19] which might cause confusion. Readers not concerned with this comparison may safely skip the present remark, which is marked by a vertical bar on the left.

In [19] an "almost surely m-to-1" extension is a factor map where almost all points in the range have *at most* m preimages. For the sake of clarity in this remark, let us call those an almost surely $(\leqslant m)$-to-1 extension. Here an "m-regular" extension is a factor map where almost all points in the range have *exactly* m preimages. Let us call those an almost surely $(= m)$-to-1 extension.

[1] This notion was introduced for the non-multivariate case of $m = 2$ in [15]. It was generalized to the multivariate case of $m > 2$ in [19].

For any factor map $\pi : (X, G) \to (Y, G)$ we have the following translation:

Result 1.1.1. "π is almost surely $(= m)$-to-1" if and only if "π is almost surely $(\leqslant m)$-to-1 *and* not almost surely $(\leqslant (m-1))$-to-1"

Sketch. The sets

$$Y_{=m} = \{y \in Y : |\pi^{-1}(\{y\})| = m\}$$
$$Y_{\leqslant m} = \{y \in Y : |\pi^{-1}(\{y\})| \leqslant m\}$$

are invariant and satisfy

$$Y_{\leqslant m} \setminus Y_{\leqslant (m-1)} = Y_{=m} .$$

By ergodicity, if $Y_{\leqslant (m-1)}$ is not of full measure it is of zero measure. Therefore if $Y_{\leqslant m}$ has full measure but $Y_{\leqslant (m-1)}$ does not we immediately conclude that $Y_{=m}$ has full measure. □

Thus the result

Result 1.1.2 ([19, Theorem 5.5 and 5.6])**.** A tds is almost surely $(\leqslant m)$-to-1 if and only if it is diam-mean $(m+1)$-equicontinuous.

immediately translates to the characterisation preceding the present remark.

We will obtain the same result under more general conditions on the group — including the (virtually) nilpotent case — as a byproduct in Section 5 (Theorem 5.2).

The following pointwise notion is natural and was already considered in [19]. A point $x \in X$ is said to be a *diam-mean m-equicontinuity point* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that (1.1) holds for all open neighborhoods U of x with $\mathrm{diam}_m(U) < \delta$. It is a direct consequence of the definitions and compactness that (X, G) is diam-mean equicontinuous if and only if all $x \in X$ are diam-mean equicontinuity points. The scheme of "almost"-equicontinuity notions applies here as well: (X, G) is called *almost diam-mean m-equicontinuous* if the set of diam-mean m-equicontinuous points is a residual subset of X. The opposite notion corresponding to diam-mean m-equicontinuity was introduced and studied by [16] for $m = 2$.[2] We generalize this to $m > 2$ and call a tds (X, G) *diam-mean m-sensitive* if there exists $\varepsilon > 0$ such that for any nonempty open subset $U \subseteq X$

$$\sup_{\mathcal{F}} \limsup_{n\to\infty} \frac{1}{|\mathcal{F}_n|} \sum_{g\in\mathcal{F}_n} \mathrm{diam}_m(gU) \;\geq\; \varepsilon \;. \tag{1.2}$$

It is then natural to define a corresponding pointwise notion: A point $x \in X$ is said to be *diam-mean m-sensitive* if there exists $\varepsilon > 0$ such that (1.2) holds for any open neighborhood U of x. Our first main result is an Auslander–Yorke type dichotomy for these notions.

Theorem 1.2 (Proven as Proposition 3.8)**.** *Suppose that (X, G) is a tds and $m \in \mathbb{N}_2$.*

(i) *If (X, G) is transitive, then it is either almost diam-mean m-equicontinuous or diam-mean m-sensitive.*
(ii) *If (X, G) is minimal, then it is either diam-mean m-equicontinuous or diam-mean m-sensitive.*

The proof will follow the standard arguments for Auslander–Yorke-type dichotomies but additionally hinges on combinatorics of diam_m. Those slightly complicate the multivariate case $m > 2$ compared to $m = 2$.

[2] See [16, Theorem 48] for the Auslander–Yorke dichotomy.

A weaker notion than that of (multivariate) diam-mean equicontinuity is that of (multivariate) frequent stability.[3] We call (X, G) *frequently m-stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\overline{\mathrm{FD}}\left(\{g \in G \mid \mathrm{diam}_m(gA) < \varepsilon\}\right) \; > \; 0 \tag{1.3}$$

holds for all sets $A \subseteq X$ with $\mathrm{diam}(A) < \delta$, where $\overline{\mathrm{FD}}(S)$ denotes the supremum of upper densities of a set $S \subseteq G$. Again, there is a dual notion corresponding to frequent m-stability. We introduce the following definition. We say that (X, G) is *strongly m-spreading* if there exists $\varepsilon > 0$ such that

$$\overline{\mathrm{FD}}\left(\{g \in G \mid \mathrm{diam}_m(gU) < \varepsilon\}\right) \; = \; 0 \tag{1.4}$$

holds for all open sets $U \subseteq X$. Similar as for diam-mean equicontinuity, frequent stability is related to a structural property of tds: in the case of minimal abelian group actions, a tds is an almost m:1-extension of its MEF if and only if it is frequently $(m+1)$-stable, but not frequently m-stable [19].

Remark 1.3. The mismatch between the notation here and in [19] – as explained in Remark 1.1 – also extends to the notion of almost m:1 extensions. Readers not concerned with this comparison may safely skip the present remark, which is marked by a vertical bar on the left.

In [19] an “almost m-to-1” extension is a factor map where a residual set of points have *at most* m preimages. Let us, for the purpose of this remark, call those an almost $(\leqslant m)$-to-1 extension. In our notation here, an “almost m:1” extension is a factor map where a residual set of points have *exactly* m preimages. Let us call those an almost $(= m)$-to-1 extension.

Let us again elaborate on the translation. Fix any factor map $\pi : (X, G) \to (Y, G)$.

Result 1.3.1. π is almost $(= m)$-to-1 if and only if π is almost $(\leqslant m)$-to-1 *and* not almost $(\leqslant (m-1))$-to-1.

Proof. Using notation from Remark 1.1 the sets $Y_{\leqslant m}$ and $Y_{=m}$ are invariant. Again,

$$Y_{\leqslant m} \setminus Y_{\leqslant (m-1)} = Y_{=m}\,.$$

By [19, Lemma 3.4], we learn that $Y_{\leqslant k}$ is G_δ for any $k \in \mathbb{N}_1$. If we assume that (X, G) is minimal, we then conclude from the invariance that

$$Y_{\leqslant (k-1)} \text{ is empty} \quad \textit{or} \quad Y_{\leqslant (k-1)} \text{ is residual}\,.$$

Therefore, if $Y_{\leqslant k}$ is residual and $Y_{\leqslant (k-1)}$ is not (and hence empty), then

$$Y_{=k} = Y_{\leqslant k} \setminus Y_{\leqslant k-1} = Y_{\leqslant k}\,.$$

And hence $Y_{=k}$ is residual. $\square$

We again consider pointwise versions of these properties: An element $x \in X$ is called a *frequent m-stability point*[4] if for all $\varepsilon > 0$ there exists $\delta > 0$ such that (1.3) holds for all $A \subseteq X$ with $x \in A$ and $\mathrm{diam}(A) < \delta$. We say x is a *strong m-spreading point* if there exists $\varepsilon > 0$ such that (1.4) holds for all open neighborhoods of x. If x is strongly m-spreading with respect to a given $\varepsilon > 0$ we will call x a *strong ε-m-spreading point.* Finally, we call (X, G) *almost frequently m-stable* if the set of frequently m-stable points is a residual subset of X. As before, we obtain an Auslander–Yorke type dichotomy.

Theorem 1.4 (Proven as Proposition 3.3)**.** *Suppose that (X, G) is a tds and $m \in \mathbb{N}_2$. Then the following hold:*

[3] This notion was introduced for the non-multivariate case in [15]. It was generalized to the multivariate case in [19].

[4] For $m = 2$ this notion was introduced and studied in [15, Definition 3.3]. It was generalized to $m > 2$ in [19, Definition 4.1].

(i) *If (X, G) is transitive, then it is either almost frequently m-stable or strongly m-spreading;*
(ii) *If (X, G) is minimal, then it is either frequently m-stable or strongly m-spreading.*

The proof proceeds along the usual Auslander–Yorke dichotomy argument, as in Theorem 1.2, and we only highlight the extra input needed to incorporate the quantity diam_m. Although the strategy is standard, the conclusion appears to be new, even when restricted to $m = 2$ (to the best of our knowledge).

Our second main objective is to provide *'local'* characterisations of the above sensitivity properties. In spirit, this is inspired by the characterisation of positive entropy via entropy tuples introduced in [7–9]. Sensitive tuples were first defined in [31] to study sensitivity in topological dynamics. In the context of mean sensitivity, the notion of mean sensitive tuples were studied [25]. Here, we provide analogue results for diam-mean equicontinuity.

We say a K-tuple $(x_k)_{k=1}^K \in X^K$ is a *weakly mean-sensitive K-tuple* [28] if for any open neighborhood U_k of x_k there exists $\eta > 0$ such that for any non-empty open subset U of X,

$$\overline{\mathrm{FD}}\left(\{g \in G : (gU) \cap U_k \neq \emptyset \text{ for } 1 \leq k \leq K\}\right) \; > \; \eta \, .$$

A tuple $x \in X^K$ is called *essential* if $x_i \neq x_j$ for all $i \neq j \in \{1, \ldots, K\}$.

Theorem 1.5 (Proven as Theorem 5.1)**.** *Suppose that (X, G) is a transitive tds and $m \in \mathbb{N}_2$. Then (X, G) is diam-mean m-sensitive if and only if there exists an essential weakly m-sensitive tuple.*

This is analoguous to the result [25, Theorem 4.4] which relates *mean m-sensitivity* with *essential m-sensitive tuples*.

A similar localised characterisation of strong m-spreading is more intricate, since it is difficult to ensure the required spreading property along a subset $S \subseteq G$ of full density with a single tuple. However, we provide an equivalent condition for strong m-spreading in terms of finite families of weakly m-sensitive tuples in Section 5.2 (see Proposition 5.6).

Structure of the article. In Section 2, we provide the necessary background and preliminaries. The Auslander–Yorke dichotomies for diam-mean m-sensitivity and diam-mean m-equicontinuity as well as for frequent m-stability and strong m-spreading are proven in Section 3 and Section 4. Section 5 then contains the characterisation of these notions in terms of weakly mean m-sensitive tuples.

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2. Preliminaries

2.1. **Notation.** Let A and B be sets. By $|A|$ we denote the cardinality of A. The symmetric difference of A and B shall be written as $A \triangle B$. If $A, B \subseteq G$ for some group G, then we let

$$A \cdot B := \{a \cdot b \,|a \in A, b \in B\} \, .$$

If (X, d) is a metric space, we will denote the open ball around $x \in X$ with radius δ by

$$B(x, \delta) := \{x' \in X \mid d(x, x') < \delta\} \, .$$

Note that we suppress the metric and space notationally. In particular this will mean, that $B(x, \delta)$ might be a ball in the original tds and $B(\pi(x), \varepsilon)$ a ball in the MEF.

2.2. **Følner Sequences and Densities.** Let G be an countably infinite and discrete group.

Definition 2.1 (Følner Sequence)**.** A sequence $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ of finite subsets $\mathcal{F}_n \subseteq G$ is called *Følner* if for any finite $K \subseteq G$ we have

$$\lim_{n\to\infty} \frac{|\mathcal{F}_n \triangle (K \cdot \mathcal{F}_n)|}{|\mathcal{F}_n|} = 0\,.$$

Definition 2.2 (Amenability)**.** We call G *amenable* if at least one Følner sequence exists.

Definition 2.3 (Tempered Følner Sequence)**.** $\mathcal{F}$ is called *tempered* if and only if there is $C > 0$ such that for all $n \in \mathbb{N}$ we have

$$\frac{\left|\bigcup_{k=1}^{n-1} \mathcal{F}_k \cdot \mathcal{F}_n\right|}{|\mathcal{F}_n|} < C\,.$$

By Proposition 1.4 in [27] every Følner sequence has a tempered subsequence.

The counting measure $A \mapsto |A|$ is clearly invariant under both left and right multiplication. Hence we have that

Proposition 2.4. *Let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ be a Følner sequence. For any $g \in G$ then the shifted sequences $g \cdot \mathcal{F} := (g \cdot \mathcal{F}_n)_{n\in\mathbb{N}}$ and $\mathcal{F} \cdot g := (\mathcal{F}_n \cdot g)_{n\in\mathbb{N}}$ are Følner.*

Proof. We prove that $\mathcal{F} \cdot g$ is Følner. So let $K \subseteq G$ be finite.

$$\frac{|K \cdot (\mathcal{F}_n \cdot g) \triangle (\mathcal{F}_n \cdot g)|}{|\mathcal{F}_n \cdot g|} = \frac{|((K \cdot \mathcal{F}_n) \cdot g) \triangle (\mathcal{F}_n \cdot g)|}{|\mathcal{F}_n|} = \frac{|K \cdot \mathcal{F}_n \triangle \mathcal{F}_n|}{|\mathcal{F}_n|} \xrightarrow[n\to\infty]{} 0.$$

The proof that $g \cdot \mathcal{F}$ is Følner is analoguous. □

From now on, we assume G to be amenable. Using Følner sequences one can define a variety of invariant averaging notions on the group G. The most simple ones of which are densities.

Definition 2.5 (Invariant density)**.** Let G be a group. A map $D : \mathcal{P}(G) \to [0,1]$ is called an *invariant density* if the following hold.

(1) Normalization: $D(\emptyset) = 0$ and $D(G) = 1$.
(2) Monotonicity: If $A \subseteq B$, then $D(A) \leq D(B)$.
(3) Invariance: For every $g \in G$ and every $A \subseteq G$,

$$D(gA) = D(A).$$

Definition 2.6. Let $A \subseteq G$ be arbitrary and fix a Følner sequence $\mathcal{F}$. We define the *upper density* of A as

$$\overline{\mathrm{D}}_{\mathcal{F}}(A) := \limsup_{n\to\infty} \frac{|A \cap \mathcal{F}_n|}{|\mathcal{F}_n|}\,.$$

Similarly, we define the *lower density* of A as

$$\underline{\mathrm{D}}_{\mathcal{F}}(A) := \liminf_{n\to\infty} \frac{|A \cap \mathcal{F}_n|}{|\mathcal{F}_n|}\,.$$

Often we want to talk about all Følner sequences at once. Let $A \subseteq G$ be arbitrary. We define

$$\overline{\mathrm{FD}}(A) := \sup_{\mathcal{F}} \limsup_{n\to\infty} \frac{|A \cap \mathcal{F}_n|}{|\mathcal{F}_n|}\,,$$

where $\sup_{\mathcal{F}}$ — as always — is the supremum taken over all Følner sequences.

Similarly, we define

$$\underline{\mathrm{FD}}(A) := \inf_{\mathcal{F}} \liminf_{n\to\infty} \frac{|A \cap \mathcal{F}_n|}{|\mathcal{F}_n|}\,,$$

where $\inf_{\mathcal{F}}$ — as always — is the infimum taken over all Følner sequences.

Lemma 2.7. *All the maps defined in Definition 2.6 are indeed invariant densities.*

Sketch. The normalization and monotonicity are obvious. The invariance uses the following consequence of the Følner property. Fix a Følner sequence $\mathcal{F}$.

$$\begin{aligned}\left|\frac{|gA\cap\mathcal{F}_n|}{|\mathcal{F}_n|}-\frac{|A\cap\mathcal{F}_n|}{|\mathcal{F}_n|}\right| &= \frac{1}{|\mathcal{F}_n|}\left||A\cap g^{-1}\mathcal{F}_n|-|A\cap\mathcal{F}_n|\right| \\ &\leqslant \frac{1}{|\mathcal{F}_n|}|g^{-1}\mathcal{F}_n\triangle\mathcal{F}_n| \xrightarrow{n\to\infty} 0\end{aligned}$$

This shows the invariance of $\overline{\mathrm{D}}_{\mathcal{F}}$ and $\underline{\mathrm{D}}_{\mathcal{F}}$. Finally, taking the supremum (resp. infimum) over all Følner sequences preserves invariance, hence $\overline{\mathrm{FD}}$ and $\underline{\mathrm{FD}}$ are invariant as well. □

2.3. **Maximal Equicontinuous Factor.** Recall that a map $p : X \to Y$ between tds (X, G) and (Y, G) is called a *factor map* if it is surjective and *equivariant*, i. e.

$$\forall g \in G : p(g(x)) = g(p(x))\,.$$

Recall that we notationally identify $g \in G$ with its corresponding homeomorphism on the state space. If $p : X \to Y$ is a factor map, we write $p : (X, G) \to (Y, G)$. A factor map $p : (X, G) \to (Y, G)$ is called an *conjugacy* if there is an inverse factor map $q : (Y, G) \to (X, G)$, i. e. $p\circ q = \mathrm{Id}_Y$, $q\circ p = \mathrm{Id}_X$.

Definition 2.8 (Equicontinuity)**.** A tds (Y, G) is called *equicontinuous*, if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $y_1, y_2 \in Y$ with $d_Y(y_1, y_2) < \delta$ we have

$$\forall g \in G : d_Y(g(y_1), g(y_2)) < \varepsilon\,,$$

where d_Y is the metric on Y.

A straightforward calculation shows that

$$d'_Y(y_1, y_2) := \sup_{g\in G} d_Y(g(y_1), g(y_2))$$

is an invariant metric that is topologically equivalent to the original metric d_Y. Thus we can — and will — always assume that metrics on equicontinuous tds are invariant.

Now let (X, G) be an arbitrary tds. Clearly, the trivial one-point tds $(\{*\}, \mathrm{Id})$ is an equicontinuous factor of (X, G). However we are interested in an equicontinuous factor which is *maximal*.

Definition 2.9 (Maximal Equicontinuous Factor)**.** A factor (Y, G) — together with the factor map $\pi : (X, G) \to (Y, G)$ — is a *maximal equicontinuous factor* (MEF) of (X, G) if (Y, G) is equicontinuous and for any factor map $p : (X, G) \to (Z, G)$ onto an equicontinuous tds (Z, G) there is a unique factor map $q : (Y, G) \to (Z, G)$ such that $p = q \circ \pi$.

$$\begin{tikzcd}
(X,G) \arrow[d, "\pi"] \arrow[dr, "p"] & \\
(Y,G) \arrow[r, dotted, "\exists!\, q"] & (Z,G)
\end{tikzcd}$$

Figure 2. Universal Property of MEF

The existence and uniqueness — up to conjugacy — of the MEF is classical:

Theorem 2.10 (see [13, Theorem 1])**.** *Any tds (X, G) has a MEF (Y, G). If (Z, G) is another MEF, then there is a conjugacy $h : (Y, G) \to (Z, G)$.*

As the MEF of (X, G) is unique, we may give it a name and call it (X_{eq}, G). The factor map to the MEF shall be denoted by $\pi_{\mathrm{eq}} : (X, G) \to (X_{\mathrm{eq}}, G)$.

2.4. **Regionally Proximal Relation.** The above description of the MEF is abstract. The proof of Theorem 2.10 in [13, Theorem 1] is more constructive but does not reveal any dynamical characterization of the equivalence relation induced by the MEF.

Such a characterization is given by the *regionally proximal relation*.

Definition 2.11 (Regionally Proximal Relation)**.** The *regionally proximal relation* $\mathcal{Q}$ is given by $(x,y) \in \mathcal{Q}$ if and only if there are sequences $(x'_n)_{n\in\mathbb{N}}, (y'_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ such that

$$\lim_{n\to\infty} x'_n = x, \quad \lim_{n\to\infty} y'_n = y \text{ and } \quad \lim_{n\to\infty} d(g_n(x'_n), g_n(y'_n)) = 0\,.$$

The link to the MEF is established by the following result from [1, Chapter 9, 3. Theorem].

Theorem 2.12. *Let (X,G) be any tds. Let $\mathcal{Q}^*$ be the smallest equivalence relation containing $\mathcal{Q}$. Then the tds $(X/\mathcal{Q}^*, G)$, together with the factor map*

$$\pi : (X,G) \longrightarrow (X/\mathcal{Q}^*, G),\ x \longmapsto \mathcal{Q}^*[x]\,,$$

is the MEF of (X,G).

Relevant for us are also the multivariate notions.

Definition 2.13 (Multivariate Regionally Proximal Relation)**.** Let $m \in \mathbb{N}_2$ and define the *regionally proximal m-relation* in the following way: $(x_1,\ldots,x_m) \in \mathcal{Q}_m$ if and only if there are sequences $(x'_{1,n})_{n\in\mathbb{N}}, \ldots, (x'_{m,n})_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ such that

$$\forall i \in \{1,\ldots,m\} : \lim_{n\to\infty} x'_{i,n} = x_i, \text{ and } \lim_{n\to\infty} \max_{i\neq j} d(g_n(x'_{i,n}), g_n(y'_{j,n})) = 0\,.$$

The dynamics inside of the classes of the regionally proximal relation is sensitive:

Lemma 2.14. *Assume that (X,G) is minimal. Then $(x_1,\ldots,x_m) \in \mathcal{Q}_m$ if and only if for any non-empty and open $B \subseteq X$ and for any collection of neighbourhoods $U_1,\ldots,U_m$ of the points $x_1,\ldots,x_m$ there is $h \in G$ such that $h(B) \cap U_i \neq \emptyset$ for $i \in \{1,\ldots m\}$.*

A proof can be found for example in [22] and [19, Lemma 2.7]. For the following rather large class of systems the multivariate regionally proximal relation is exactly given by the pairwise regionally proximal relation and the regionally proximal relation is already a (closed) equivalence relation.

Definition 2.15 (Locally Bronstein)**.** A system is called *locally Bronstein* if for each pair $(x,y) \in X^2$ which is almost periodic in (X^2,G) and satisfies $(x,y) \in \mathcal{Q}$ there are sequences $(x''_n)_{n\in\mathbb{N}}, (y''_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ with (x''_n, y''_n) almost periodic in (X^2,G) and there is a sequence $(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ such that

$$\lim_{n\to\infty} x''_n = x, \quad \lim_{n\to\infty} y''_n = y \text{ and } \quad \lim_{n\to\infty} d(g_n(x''_n), g_n(y''_n)) = 0$$

for all $i \in \{1,\ldots,m\}$. In other words, the approximating points $(x''_n)_{n\in\mathbb{N}}, (y''_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ in Definition 2.11 can be chosen almost periodic.

For many acting groups the local Bronstein property is satisfied automatically.

Lemma 2.16. *If the acting group G is virtually nilpotent — in particular, abelian — any minimal system (X,G) is incontractible, i. e. the almost periodic points of (X^n,G) are dense in X^n for any $n \in \mathbb{N}$.*

Remark 2.17. a) A straightforward argument (just using the case $n = 2$) shows that any incontractible system is locally Bronstein.

b) The main source of Lemma 2.16 is (the proof of) [18, Theorem 7.5]. However, it is not immediately clear how to connect the proof and statement of [18, Theorem 7.5] with Lemma 2.16. The way to bridge that gap is spread across the Literature. For ease of the reader and precise referencing we sketch how that connection may be done:

In [17, Definition 4.3] the above definition of *incontractible systems* is given. Minimal incontractible systems are characterized by disjointness to any minimal proximal flow [17, Proposition 4.4].[5] In the proof of [18, Theorem 7.5], we now see that for virtually nilpotent acting groups the minimal proximal systems are all trivial. Thus the condition of being disjoint to all of them is vacuously satisfied. Hence all systems with a virtually nilpotent acting group are incontractible.

Now [3, Theorem 8] states the following fact:

Theorem 2.18. *Assume that (X, G) is locally Bronstein. For any $m \in \mathbb{N}_2$ and any $(x_1, \ldots, x_m) \in X^m$ we have*

$$(x_1, \ldots, x_m) \in \mathcal{Q}_m \quad \text{if and only if} \quad (x_1, x_i) \in \mathcal{Q} \text{ for any } i \in \{1, \ldots, m\}.$$

Furthermore, $\mathcal{Q}$ is an equivalence relation.

Remark 2.19. This is not the exact statement of [3, Theorem 8]. However combining lemmas from [2] and [3] one obtains Theorem 2.18.

3. Auslander–Yorke type dichotomies for multivariate diam-mean equicontinuity and frequent stability

The arguments for Auslander–Yorke type dichotomies usually follow a similar structure: The set of ε-sensitivity points — for each sensitivity constant $\varepsilon > 0$ — is invariant and closed. Therefore, if a transitivity point is sensitive — and thus ε-sensitive for some $\varepsilon > 0$ — the whole system is uniformly ε-sensitive – and hence sensitive.

Our proofs follow this standard structure utilizing an understanding of the properties of D_m and diam_m as well as the multivariate definitions.

3.1. **Multivariate frequent stability and strong spreading.** Let (X, G) be a tds and $m \in \mathbb{N}_2$. Following ideas in [24], we study the set $\mathcal{FS}^m$ of frequently m-stability points. For every $\varepsilon > 0$, let

$$\mathcal{FS}^m_\varepsilon \;:=\; \left\{ x \in X \;\middle|\; \exists \delta > 0 : \overline{\mathrm{FD}}\left(\{g \in G \mid \mathrm{diam}_m(gB(x,\delta)) < \varepsilon\}\right) \;>\; 0 \right\}. \tag{3.1}$$

Note that $(\mathcal{FS}^m_\varepsilon)^c$ is the set of strongly ε-m-spreading points.

Proposition 3.1. *Let $\varepsilon > 0$. Then $\mathcal{FS}^m_\varepsilon$ is open and invariant, i. e. $h(\mathcal{FS}^m_\varepsilon) = \mathcal{FS}^m_\varepsilon$ for any $h \in G$.*

Proof. Let $x \in \mathcal{FS}^m_\varepsilon$. Choose $\delta > 0$ — as in the definition of $\mathcal{FS}^m_\varepsilon$ in Equation (3.1) — such that

$$\overline{\mathrm{FD}}\left(\{g \in G \mid \mathrm{diam}_m(gB(x,\delta)) < \varepsilon\}\right) \;>\; 0.$$

Note that for any $y \in B(x, \frac{\delta}{2})$, we have $B(y, \frac{\delta}{2}) \subseteq B(x, \delta)$. By monotony of diam_m, we obtain

$$\overline{\mathrm{FD}}\left(\left\{g \in G \mid \mathrm{diam}_m\left(gB\left(y, \tfrac{\delta}{2}\right)\right) < \varepsilon\right\}\right) \;>\; 0.$$

Thus $y \in \mathcal{FS}^m_\varepsilon$, and hence $\mathcal{FS}^m_\varepsilon$ is open.

In order to show the invariance, let $h \in G$ be arbitrary and assume that $x \in \mathcal{FS}^m_\varepsilon$. We will show $h^{-1}x \in \mathcal{FS}^m_\varepsilon$. Let $\delta > 0$ as in the definition of $\mathcal{FS}^m_\varepsilon$ in Equation (3.1). As $h : X \to X$ is continuous, for any $\delta > 0$ there exists $\eta_\delta > 0$ such that $hB(h^{-1}x, \eta_\delta) \subseteq B(x, \delta)$.

By assumption,

$$\overline{\mathrm{FD}}\left(\{g \in G \mid \mathrm{diam}_m\left(gB\left(x,\delta\right)\right) < \varepsilon\}\right) > 0.$$

By Lemma 2.7, $\overline{\mathrm{FD}}$ is invariant. We conclude

$$\overline{\mathrm{FD}}\left(\left\{g \in G \mid \mathrm{diam}_m\left(gB\left(h^{-1}x, \eta_\delta\right)\right) < \varepsilon\right\}\right)$$

[5] For a minimal system *incontractibility* is equivalent to *un-contractibility* — in the sense of [18]. This is due to the characterization in [18, Theorem 4.2] which states that for minimal systems non-contractibility is again equivalent to disjointness to any minimal proximal flow.

$$\begin{aligned}
&= \overline{\mathrm{FD}}\left(\left\{g' \in G \mid \operatorname{diam}_m\left(g'hB\left(h^{-1}x, \eta_\delta\right)\right) < \varepsilon\right\}\right) \\
&\geqslant \overline{\mathrm{FD}}\left(\left\{g' \in G \mid \operatorname{diam}_m\left(g'B\left(x, \delta\right)\right) < \varepsilon\right\}\right) \\
&> 0\,.
\end{aligned}$$

We conclude $h^{-1}x \in \mathcal{FS}^m_\varepsilon$. As $h \in G$ was arbitrary, this proves the invariance of $\mathcal{FS}^m_\varepsilon$. □

Remark 3.2. Consequently, the set of strong ε-m-spreading points is closed and invariant. Furthermore, as $\mathcal{FS}^m = \cap_{n=1}^\infty \mathcal{FS}^m_{\varepsilon_n}$ for any $\varepsilon_n \xrightarrow{n\to\infty} 0$, we conclude that $\mathcal{FS}^m$ is G_δ.

This allows us to restate and prove Theorem 1.4.

Proposition 3.3 (Auslander–Yorke Dichotomy for Frequent Stability)**.** *Let (X, G) be a transitive tds. Then (X, G) is either almost frequently m-stable — i. e. $\mathcal{FS}^m$ is a dense G_δ-set — or (X, G) is strongly m-spreading — i. e. $\mathcal{FS}^m = \emptyset$.*

If in addition, (X, G) is minimal, then either (X, G) is frequently m-stable — i. e. $\mathcal{FS}^m = X$ — or (X, G) is strongly m-spreading — i. e. $\mathcal{FS}^m = \emptyset$.

Proof. It is clear, that the cases of the dichotomy are disjoint. Let (X, G) be transitive and let x be a transitivity point, i. e. the orbit of x is dense.

Assume on the one hand that $x \in \mathcal{FS}^m$. Then, by invariance, $\mathcal{FS}^m$ contains the orbit of x and is thus dense. By Remark 3.2, $\mathcal{FS}^m$ is furthermore a G_δ-set. Hence $\mathcal{FS}^m$ is dense G_δ and hence (X, G) is almost frequently m-stable.

Assume on the other hand that $x \notin \mathcal{FS}^m$. Then there is $\varepsilon > 0$ such that $x \in (\mathcal{FS}^m_\varepsilon)^c$. As above, by invariance, $(\mathcal{FS}^m_\varepsilon)^c$ is dense. Finally, by the closedness of $(\mathcal{FS}^m_\varepsilon)^c$, we have $X = (\mathcal{FS}^m_\varepsilon)^c$. Therefore (X, G) is strongly m-spreading.

Let (X, G) now be minimal. Then any $x \in X$ is a transitivity point. Assume that there is $\varepsilon > 0$ and $y \in X$ such that $y \in (\mathcal{FS}^m_\varepsilon)^c$. Then again — as y has dense orbit — $X = (\mathcal{FS}^m_\varepsilon)^c$. Hence $\mathcal{FS}^m = \emptyset$. □

Remark 3.4. Note that the proof of Proposition 3.1 only uses the fact that $\overline{\mathrm{FD}}$ is an invariant density as defined by Definition 2.5. Moreover, Proposition 3.3 follows directly from Proposition 3.1. Hence the Auslander–Yorke Dichotomy also holds for all the frequent m-stability-like notions obtained by replacing $\overline{\mathrm{FD}}$ in the definition by any other invariant density. This shows that the "strong" and "weak frequent m-stability" defined in [19, Definition 4.1] are also subject to an analogous Auslander–Yorke Dichotomy.

3.2. **Multivariate diam-mean equicontinuity and sensitivity.** Again, let (X, G) be a tds and $m \in \mathbb{N}_2$. Following ideas in [24], we study the set of diam-mean m-equicontinuous points, denoted by $\mathcal{DE}^m$. For every $\varepsilon > 0$, let

$$\mathcal{DE}'^m_\varepsilon := \left\{ x \in X \;\middle|\; \exists \delta > 0 : \sup_{\mathcal{F}} \limsup_{n\to\infty} \frac{1}{|\mathcal{F}_n|} \sum_{g \in \mathcal{F}_n} \operatorname{diam}_m(gB(x,\delta)) < \varepsilon \right\}. \tag{3.2}$$

We also define a pointwise notion which is more analoguous to (3.1) by

$$\mathcal{DE}^m_\varepsilon := \left\{ x \in X \;\middle|\; \exists \delta > 0 : \overline{\mathrm{FD}}\left(\left\{g \in G \mid \operatorname{diam}_m(gB(x,\delta)) \leqslant \varepsilon\right\}\right) > 1 - \varepsilon \right\}. \tag{3.3}$$

Note that $(\mathcal{DE}^m_\varepsilon)^c$ is the set of diam-mean ε-m-sensitivity points. It is in general not true that $\mathcal{DE}'^m_\varepsilon = \mathcal{DE}^m_\varepsilon$. Nevertheless the following follows from (the proof of) [19, Lemma 5.2]:

Lemma 3.5. *If we normalize $\operatorname{diam}_m(X) = 1$, then*

$$\mathcal{DE}^m_{\frac{\varepsilon}{2}} \subseteq \mathcal{DE}'^m_\varepsilon \subseteq \mathcal{DE}^m_{\sqrt{\varepsilon}}.$$

It follows that

$$\mathcal{DE}^m = \bigcap_{\varepsilon>0} \mathcal{DE}'^m_\varepsilon = \bigcap_{\varepsilon>0} \mathcal{DE}^m_\varepsilon .$$

Those facts allow the following two propositions and their proofs to be presented completely analogously to the corresponding statements from the previous Subsection 3.1. We still state and prove them for the sake of convenience of the reader.

Proposition 3.6. *Let $\varepsilon > 0$. Then $\mathcal{DE}^m_\varepsilon$ is open and invariant, i. e. $h(\mathcal{DE}^m_\varepsilon) = \mathcal{DE}^m_\varepsilon$ for any $h \in G$.*

Proof. For any $x \in \mathcal{DE}^m_\varepsilon$, we choose $\delta > 0$ — as in the definition of $\mathcal{DE}^m_\varepsilon$ in Equation (3.3) — such that

$$\overline{\mathrm{FD}}\left(\{g \in G \mid \mathrm{diam}_m(gB(x,\delta)) \leqslant \varepsilon\}\right) > 1-\varepsilon .$$

Note that for any $y \in B(x, \frac{\delta}{2})$, we have $B(y, \frac{\delta}{2}) \subseteq B(x,\delta)$. By monotony of diam_m, we obtain

$$\overline{\mathrm{FD}}\left(\left\{g \in G \mid \mathrm{diam}_m\left(gB\left(y,\tfrac{\delta}{2}\right)\right) \leqslant \varepsilon\right\}\right) > 1-\varepsilon .$$

Thus $y \in \mathcal{DE}^m_\varepsilon$, and hence $\mathcal{DE}^m_\varepsilon$ is open.

In order to show the invariance, let $h \in G$ be arbitrary and assume that $x \in \mathcal{DE}^m_\varepsilon$. We will show $h^{-1}x \in \mathcal{DE}^m_\varepsilon$. Let $\delta > 0$ as in the definition of $\mathcal{DE}^m_\varepsilon$ in Equation (3.3). As $h : X \to X$ is continuous, for any $\delta > 0$ there exists $\eta_\delta > 0$ such that $hB(h^{-1}x, \eta_\delta) \subseteq B(x,\delta)$.

By assumption,

$$\overline{\mathrm{FD}}\left(\{g \in G \mid \mathrm{diam}_m\left(gB\left(x,\delta\right)\right) \leqslant \varepsilon\}\right) > 1-\varepsilon .$$

By Lemma 2.7, $\overline{\mathrm{FD}}$ is invariant. We conclude

$$\begin{aligned} &\overline{\mathrm{FD}}\left(\left\{g \in G \mid \mathrm{diam}_m\left(gB\left(h^{-1}x,\eta_\delta\right)\right) \leqslant \varepsilon\right\}\right)\\ =&\overline{\mathrm{FD}}\left(\left\{g' \in G \mid \mathrm{diam}_m\left(g'hB\left(h^{-1}x,\eta_\delta\right)\right) \leqslant \varepsilon\right\}\right)\\ \geqslant&\overline{\mathrm{FD}}\left(\{g' \in G \mid \mathrm{diam}_m\left(g'B\left(x,\delta\right)\right) \leqslant \varepsilon\}\right)\\ >&1-\varepsilon . \end{aligned}$$

We conclude $h^{-1}x \in \mathcal{DE}^m_\varepsilon$. As $h \in G$ was arbitrary, this proves the invariance of $\mathcal{DE}^m_\varepsilon$. ☐

Remark 3.7. Consequently, the set of diam-mean ε-m-sensitivity points is closed and invariant. Furthermore, as $\mathcal{DE}^m = \cap_{n=1}^\infty \mathcal{DE}^m_{\varepsilon_n}$ for any $\varepsilon_n \xrightarrow{n\to\infty} 0$, we conclude that $\mathcal{DE}^m$ is G_δ.

We thus can reformulate and prove Theorem 1.2:

Proposition 3.8 (Auslander–Yorke Dichotomy for Diam-Mean Equicontinuity)**.** *Let (X,G) be a transitive tds. Then (X,G) is either almost diam-mean m-equicontinuous — i. e. $\mathcal{DE}^m$ is a dense G_δ-set — or (X,G) is diam-mean m-sensitive — i. e. $\mathcal{DE}^m = \emptyset$.*

If in addition, (X,G) is minimal, then either (X,G) is diam-mean m-equicontinuous — i. e. $\mathcal{DE}^m = X$ — or (X,G) is diam-mean m-sensitive — i. e. $\mathcal{DE}^m = \emptyset$.

Proof. It is clear, that the cases of the dichotomy are disjoint. Let (X,G) be transitive with a transitive point $x \in X$.

Assume on the one hand that $x \in \mathcal{DE}^m$. Then $\mathcal{DE}^m$ is a dense and by Remark 3.7, $\mathcal{DE}^m$ is a G_δ-set. Thus $\mathcal{DE}^m$ is dense G_δ and hence (X,G) is almost diam-mean m-equicontinuous.

Assume on the other hand that $x \notin \mathcal{DE}^m$. Then there is $\varepsilon > 0$ such that $x \in (\mathcal{DE}^m_\varepsilon)^c$. By the invariance of $(\mathcal{DE}^m_\varepsilon)^c$, we have $Gx \subseteq (\mathcal{DE}^m_\varepsilon)^c$. Finally, by the closedness, we have $X = \overline{Gx} \subseteq (\mathcal{DE}^m_\varepsilon)^c$. Therefore (X,G) is diam-mean m-sensitive.

Let (X,G) now be minimal. Then any $x \in X$ is a transitivity point. Assume that there is $\varepsilon > 0$ and $y \in X$ such that $y \in (\mathcal{DE}^m_\varepsilon)^c$. Then again — as y has dense orbit — so $X = (\mathcal{DE}^m_\varepsilon)^c$. So $\mathcal{DE}^m = \emptyset$. ☐

Remark 3.9. As in the case for frequent m-stability – discussed in Remark 3.4 – the arguments for the preceding propositions work verbatim if $\overline{\mathrm{FD}}$ is replaced by any other invariant density. Hence, we can conclude that also other diam-mean m-equicontinuity notions, where other invariant densities are used in the definitions, are subject to analogous Auslander–Yorke Dichotomies. This in particular applies to the notions of "strong" and "weak diam-mean m-equicontinuity" defined in [19, Definition 5.1].

4. Multivariate diam-mean equicontinuity and multivariate regularity

The following result is proven in [19, Theorem 5.1, (i) implies (ii)] under the assumption of locally Bronstein dynamics and – as mentioned in Remark 3.4 – using a different invariant density in the definition of diam-mean $(m+1)$-equicontinuity. Still, the same structural argument applies to our case once the hypotheses and definitions are adjusted.

Therefore, we have the following relation between diam-mean m-equicontinuity and m-regularity.

Theorem 4.1. *Let (X,G) be a minimal tds and $m \in \mathbb{N}_1$. If π_{eq} is regular m-to-one, then (X,G) is diam-mean $(m+1)$-equicontinuous.*

Since the statement is not an immediate citation but a consequence of the proof in [19, Theorem 5.1, (i) implies (ii)], we reproduce the relevant argument adapted to our setting. The proof is formally identical. The only changes concern the assumptions and the underlying definitions.

Repetition of proof "(i) ⇒ (ii)" [19, Theorem 5.5]. By the above, we only need to prove that any $x \in X$ is a diam-mean m-equicontinuous point. For convenience, we assume without loss of generality that $\mathrm{diam}(X) = 1$.

Let $\varepsilon > 0$. By assumption π_{eq} is m-regular, so – by inner regularity of the measure μ_{eq} – we can choose a compact subset K of X_{eq} with $\mu_{\mathrm{eq}}(K) > 1 - \frac{\varepsilon}{3}$ and $|\pi_{\mathrm{eq}}^{-1}(y)| = m$ for any $y \in K$. Since π_{eq}^{-1} is upper semi-continuous, there exists $\delta > 0$ and $z_1, \dots z_l \in K$ such that $K \subseteq \cup_{i=1}^{l} B(z_i, \delta)$ and

$$\pi_{\mathrm{eq}}^{-1}(B(z_i, 2\delta)) \subseteq B(\pi_{\mathrm{eq}}^{-1}(\{z_i\}), \tfrac{\varepsilon}{3}) .$$

Now we choose $\eta > 0$ such that $\mathrm{diam}(\pi_{\mathrm{eq}}(B(x,\eta))) < \delta$ holds for any $x \in X$. Notice that for any $x \in X$, if $g \in G$ such that $g\pi_{\mathrm{eq}}(x) = \pi_{\mathrm{eq}}(gx) \in K$, there exists $j \in \{1, 2, \dots, l\}$ such that $\pi_{\mathrm{eq}}(gx) \in B(z_i, \delta)$. Since $\mathrm{diam}(g\pi_{\mathrm{eq}}(B(x,\eta))) = \mathrm{diam}(\pi_{\mathrm{eq}}(B(x,\eta))) < \delta$, it follows that $\pi_{\mathrm{eq}}(gB(x,\eta)) \subseteq B(z_i, 2\delta)$,. This further implies that

$$gB(x,\eta) \;\subseteq\; \pi_{\mathrm{eq}}^{-1}(B(z_i, 2\delta)) \;\subseteq\; B(\pi_{\mathrm{eq}}^{-1}(z_i), \tfrac{\varepsilon}{3}) \; .$$

As $|\pi_{\mathrm{eq}}^{-1}(z_i)| = m$, we conclude $\mathrm{diam}_{m+1}(gB(x,\eta)) < \frac{2\varepsilon}{3}$ from the definition of diam_{m+1}.[6] Thus, we deduce that

$$\limsup_{n\to\infty} \frac{1}{|\mathcal{F}_n|} \sum_{g \in \mathcal{F}_n} \mathrm{diam}_{m+1}(gB(x,\eta))$$

$$(4.1) = \quad \limsup_{n\to\infty} \frac{1}{|\mathcal{F}_n|} \left(\sum_{g \in \mathcal{F}_n, \pi_{\mathrm{eq}}(gx) \in K} \mathrm{diam}_{m+1}(gB(x,\eta)) + \sum_{g \in \mathcal{F}_n, \pi_{\mathrm{eq}}(gx) \notin K} \mathrm{diam}_{m+1}(gB(x,\eta)) \right)$$

---

[6] By definition

$$\mathrm{diam}_{m+1}\left(B(\pi_{\mathrm{eq}}^{-1}(z_i), \tfrac{\varepsilon}{3})\right) = \max \left\{ \min_{i \neq j} d(a_i, a_j) : (a_1, \dots, a_{m+1}) \in B(\pi_{\mathrm{eq}}^{-1}(z_i), \tfrac{\varepsilon}{3})^{m+1} \right\} .$$

For a fixed $(a_1, \dots, a_{m+1}) \in B(\pi_{\mathrm{eq}}^{-1}(z_i), \frac{\varepsilon}{3})^{m+1}$ and $b_i \in B$ and furthermore $D_m(a_1, \dots, a_{m+1}) = d(a_{i_a}, a_{j_a})$ for some $i_a, j_a \in \{1, \dots, m+1\}$. Now for each $(a_1, \dots, a_{m+1}) \in B(\pi_{\mathrm{eq}}^{-1}(z_i), \frac{\varepsilon}{3})^{m+1}$ there are $(b_1, \dots, b_{m+1}) \in \pi_{\mathrm{eq}}^{-1}(z_i)^{m+1}$ with $a_k \in B(b_k, \frac{\varepsilon}{3})$. So

$$D_m(a_1, \dots, a_{m+1})) \leqslant d(a_{i_b}, a_{j_b}) \leqslant d(b_{i_a}, b_{j_a}) + \tfrac{\varepsilon}{3} = D_m(b_1, \dots, b_{m+1}) + \tfrac{\varepsilon}{3} .$$

This inequality now carries over to the maximum and thus to diam_{m+1}.

$$= \frac{2\varepsilon}{3} \cdot \limsup_{n\to\infty} \frac{1}{|\mathcal{F}_n|} |\{g \in \mathcal{F}_n : g\pi_{\mathrm{eq}}(x) \in K\}| + \operatorname{diam}(X) \cdot \limsup_{n\to\infty} \frac{1}{|\mathcal{F}_n|} |\{g \in \mathcal{F}_n : g\pi_{\mathrm{eq}}(x) \notin K\}| \; .$$

Since (X_{eq}, G) is uniquely ergodic, by the Birkhoff Ergodic Theorem (by Lindenstrauss [27]), for any tempered Følner sequence $\mathcal{F} = \{\mathcal{F}_n\}_{n\in\mathbb{N}}$, (4.1) implies that

$$\limsup_{n\to\infty} \frac{1}{|\mathcal{F}_n|} \sum_{g\in\mathcal{F}_n} \operatorname{diam}_{m+1}(gB(x,\eta)) \leq 2\varepsilon \cdot \mu_{\mathrm{eq}}(K) + (1-\mu_{\mathrm{eq}}(K)) < \varepsilon \; .$$

Recall that any Følner sequence has a tempered Følner subsequence [27, Proposition 1.4], this completes the proof. □

In Section 5, we also prove the converse under the assumption of a local Bronstein condition. In the abelian case those results are a consequence of [19, Theorem 5.6].[7]

We now turn to the notion of diam-mean sensitivity. A direct consequence of Proposition 3.6 is the following:

Proposition 4.2. *Let (X, G) be a tds with a transitive point x and $m \in \mathbb{N}_2$. If x is diam-mean m-sensitive then (X, G) is diam-mean m-sensitive.*

For the convenience of the reader we provide an alternative proof.

Alternative Proof. Since x is diam-mean m-sensitive, there exists $\varepsilon > 0$ such that for any open neighborhood U of x,

$$\sup_{\mathcal{F}} \limsup_{n\to\infty} \frac{1}{|\mathcal{F}_n|} \sum_{g\in\mathcal{F}_n} \operatorname{diam}_m(gU) \;\geq\; \varepsilon \; .$$

Fix a nonempty open subset V of X. As x is transitive, there exists $h \in G$ such that $hx \in V$. By continuity of h, we can find an open neighborhood U of x such that $hU \subseteq V$. Thus, we have

$$\sup_{\mathcal{F}} \limsup_{n\to\infty} \frac{1}{|\mathcal{F}_n|} \sum_{g\in\mathcal{F}_n} \operatorname{diam}_m(gV) \;\geq\; \sup_{\mathcal{F}} \limsup_{n\to\infty} \frac{1}{|\mathcal{F}_n h|} \sum_{g\in\mathcal{F}_n h} \operatorname{diam}_m(gU) \;>\; \varepsilon \, .$$

This completes the proof. □

5. Weakly mean sensitive tuples

5.1. **Multivariate diam-mean sensitivity and weakly mean sensitive tuples.** Following ideas in [25, Theorem 4.4], we can now formulate and prove 1.5:

[7] Recall Remark 1.1 for translating the results. Furthermore, observe that the notions of diam-mean m-equicontinuity slightly vary between the papers, as different averages are considered. We here focus on the density $\overline{\mathrm{FD}}$ defined in Definition 2.6. Recall that $\underline{\mathrm{BD}}_{\mathcal{F}}$ and $\overline{\mathrm{BD}}_{\mathcal{F}}$ denote the lower and upper Banach densities along a Følner sequence $\mathcal{F}$, as in [19, Definition 1.10]. Then, by definition,

$$\underline{\mathrm{BD}} := \sup_{\mathcal{F}} \underline{\mathrm{BD}}_{\mathcal{F}} \;\leq\; \underline{\mathrm{FD}} \;\leq\; \overline{\mathrm{FD}} \;\leq\; \overline{\mathrm{BD}} := \sup_{\mathcal{F}} \overline{\mathrm{BD}}_{\mathcal{F}} \, .$$

Note that [19, Theorem 5.6] also entails that the two densities $\underline{\mathrm{BD}}$ and $\overline{\mathrm{BD}}$ lead to equivalent notions of diam-mean m-equicontinuity. This further implies that also any densities in between, e. g. $\overline{\mathrm{FD}}$, yields the same diam-mean m-equicontinuity notion.

In the non-abelian there is no such result in [19]. However, for a $\overline{\mathrm{BD}}$ [19, Theorem 5.5] states that the corresponding notion of diam-mean equicontinuity is equivalent to (again considering the necessary translation) almost-sure m-to-1-ness. This can be read productively as entailing the equivalence of the diam-mean equicontinuity notion corresponding to $\overline{\mathrm{BD}}$ and $\overline{\mathrm{FD}}$.

Theorem 5.1. *Let (X,G) be a tds and $m \in \mathbb{N}_2$. If (X,G) has an essential weakly mean-sensitive m-tuple, then (X,G) is diam-mean m-sensitive. If we further assume that (X,G) is transitive, then the converse direction also holds.*

Proof. First, we assume that $(x_1,\dots,x_m)$ is an essential weakly mean-sensitive m-tuple. Let $\delta := \frac{1}{2}\min_{1\le i<j\le m} d(x_i,x_j) > 0$. For each $k \in \{1,2,\dots,m\}$, we choose a neighborhood U_k of x_k by $U_k := B(x,\frac{\delta}{2})$. Therefore we obviously have

$$\min_{1\le i<j\le m} d(U_i,U_j) > \delta. \tag{5.1}$$

Then there exists $\eta > 0$ such that for any open subset U of X,

$$\overline{\mathrm{FD}}(\{g\in G : (gU)\cap U_k \neq \emptyset \text{ for } 1\le k\le m\}) > \eta.$$

Let

$$\mathcal{N} := \{g\in G : (gU)\cap U_k \neq \emptyset \text{ for } 1\le k\le m\}.$$

Then it is easy to see that for any $g\in\mathcal{N}$, $\mathrm{diam}_m(gU) > \delta$. Thus,

$$\sup_{\mathcal{F}} \limsup_{n\to\infty} \frac{1}{|\mathcal{F}_n|}\sum_{g\in\mathcal{F}_n} \mathrm{diam}_m(gU) \ > \ \delta\eta\ .$$

Therefore, (X,G) is diam-mean m-sensitive with constant $\varepsilon = \delta\eta$.

In order to prove the converse, we suppose that (X,G) is a transitive diam-mean m-sensitive tds. By Lemma 3.5, there exists $\varepsilon > 0$ such that for any nonempty open set U of X, there exists a subset F of G with $\overline{\mathrm{FD}}(F)\ge\varepsilon$ such that for any $g\in F$,

$$\mathrm{diam}_m(gU) \ \ge\ \varepsilon\ .$$

This implies that for any $f\in F$ there exist $x_1^{(f)},\dots,x_m^{(f)}\in U$ such that

$$\min_{1\le i<j\le m} d(fx_i^{(f)}, fx_j^{(f)}) \ge \varepsilon. \tag{5.2}$$

Put

$$X_\varepsilon^m \ :=\ \{(x_1,\dots,x_m)\in X^m : \min_{1\le i<j\le m} d(x_i,x_j)\ge\varepsilon\}\ .$$

Then X_ε^m is a compact subset of X^m. Fix a transitive point $x\in X$ and set $W_n := B(x,\frac{1}{n})$ for each $n\in\mathbb{N}$. By (5.2), for each $n\in\mathbb{N}$,

$$\overline{\mathrm{FD}}(\{g\in G : (gW_n)^m\cap X_\varepsilon^m \neq\emptyset\}) \ \ge\ \varepsilon\ .$$

Due to the compactness of X_ε^m, there exists a finite family of open sets $\{A_i^{(1)}\}_{i=1}^{N_1}$ with $\mathrm{diam}(A_i^{(1)})\le 1$ such that

$$X_\varepsilon^m \ \subseteq\ \cup_{i=1}^{N_1} A_i^{(1)}\ .$$

By the pigeonhole principle, we may assume that for any $n\in\mathbb{N}$, there exists $p_n\in\{1,2,\dots,N_1\}$ such that

$$\overline{\mathrm{FD}}(\{g\in G : (gW_n)^m\cap A_{p_n}^{(1)}\cap X_\varepsilon^m \neq\emptyset\}) \ge \tfrac{\varepsilon}{N_1}\ .$$

Since the sets W_n are decreasing and $N_1<\infty$, we may choose the same p_n for all $n\in\mathbb{N}$. We can thus assume without loss of generality that $p_n = 1$ for all $n\in\mathbb{N}$.

Repeating the produce above, for each $l\ge 1$, we may assume that there exists an open subset $A^{(l)}$ with $\mathrm{diam}(A^{(l)}) < \frac{1}{l}$ such that $A^{(l)}\subseteq A^{(l-1)}$ and for all $n\in\mathbb{N}$,

$$\overline{\mathrm{FD}}\left(\left\{g\in G : (gW_n)^m\cap A^{(l)}\cap X_\varepsilon^m\neq\emptyset\right\}\right) \ \ge\ \frac{\varepsilon}{N_1N_2\dots N_l}\ . \tag{5.3}$$

As $\lim_{l\to\infty}\mathrm{diam}(A^{(l)}) = 0$ and the space is compact, it is clear that $(\cap_{l=1}^\infty A^{(l)})\cap X_\varepsilon^m$ is a singleton, denoted by $(x_1,\dots,x_m)$. Now we prove this point is a weakly mean-sensitive tuple. Indeed, for any family of open neighborhoods U_i of x_i, $i = 1, 2, \ldots, m$, there exists $l > 0$ such that $A^{(l)} \cap X_\varepsilon^m \subseteq U_1 \times \ldots \times U_m$. By (5.3), for all $n \in \mathbb{N}$,

$$\overline{\mathrm{FD}}\left(\{g \in G : (gW_n)^m \cap (U_1 \times \ldots \times U_m) \neq \emptyset\}\right) \;\geq\; \frac{\varepsilon}{N_1 N_2 \ldots N_l}\;.$$

For any open subset U of X, as $x \in X$ is a transitive point, there exists $h \in G$ such that $hx \in U$. By the continuity of h, we can choose $n \in \mathbb{N}$ large enough such that $hW_n \subseteq U$. Then we have

$$\overline{\mathrm{FD}}\left(\{g \in G : (gU) \cap U_i \neq \emptyset, \text{ for } i = 1, 2, \ldots, m\}\right) \;\geq\; \frac{\varepsilon}{N_1 N_2 \ldots N_l}\;.$$

This completes the proof. □

In order to prove a converse to Theorem 4.1, we need an additional condition that is required in order to apply results from [28]. Recall that a tds (X, G) satisfies the *local Bronstein condition* if for any minimal point $(x, x') \in X \times X$ such that $\pi_{\mathrm{eq}}(x) = \pi_{\mathrm{eq}}(x')$ there exists a sequence $((x_n, x'_n))_{n\in\mathbb{N}}$ of minimal points in $X \times X$ and a sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $\lim_{n\to\infty}(x_n, x'_n) = (x, x')$ and $\lim_{n\to\infty} d(g_n x, g_n x') = 0$. Note that this is always satisfied if (X, G) is a minimal action of an abelian group (Lemma 2.16). The following statement now summarises our findings combined with the respective results in [28].

Theorem 5.2. *Suppose that (X, G) is a minimal tds that satisfies the local Bronstein condition. Let $m \in \mathbb{N}_1$. Then the following statements are equivalent:*

- *(i) π_{eq} is regular m-to-one;*
- *(ii) there exists an essential weakly m-sensitive tuple and no essential weakly $(m + 1)$-sensitive tuple;*
- *(iii) for every $y \in X_{\mathrm{eq}}$, $\pi_{\mathrm{eq}}^{-1}(y)$ contains an essential weakly m-sensitive tuple and no essential weakly $(m + 1)$-sensitive tuples;*
- *(iv) (X, G) is diam-mean $(m + 1)$-equicontinuous but not diam-mean m-equicontinuous;*
- *(v) (X, G) is not diam-mean $(m + 1)$-sensitive, but diam-mean m-sensitive.*

Proof. (i) ⇒ (iii). The first statement has been proven in [28, Corollary 5.3]. Now, assume for a contradiction that there exists an essential weakly $(m + 1)$-sensitive tuple. By Theorem 5.1, we know that (X, G) is diam-mean $(m + 1)$-sensitive. Hence Proposition 3.8 implies that (X, G) is not diam-mean $(m + 1)$-equicontinuous. However, this contradicts Theorem 4.1.

(iii) ⇒ (ii) is trivial.

(ii) ⇒ (i). By Corollary 5.3 in [28], as (X, G) has no essential weakly $(m + 1)$-sensitive tuple, we know that π_{eq} is regular m'-to-one for some $0 < m' \leq m$. As (X, G) has an essential weakly m-sensitive tuple, the fact that (i) ⇒ (iii) yields $m' \geq m$. Therefore (i) holds.

(i) ⇒ (iv) has been proven in Theorem 4.1.

(iv) ⇔ (v) holds by Proposition 3.8.

(v) ⇒ (ii) finally follows from Theorem 5.1. □

Remark 5.3. As a consequence we can elaborate a bit more on Remark 1.1. Recall the notation used there and that we have that π_{eq} is almost-surely $(= m)$-to-1 (i. e. m-regular) if and only if it is almost-surely $(\leqslant m)$-to-1 and not almost-surely $(\leqslant m)$-to-one. By Theorem 5.2 we conclude that π_{eq} is almost-surely $(= m)$-to-1 (i. e. m-regular) if and only if π_{eq} is almost-surely $(\leqslant m)$-to-1 and (X, G) is diam-mean m-sensitive.

5.2. **Strong m-spreading and mean sensitive tuples.** For the sake of completeness, we discuss in this section how strong m-spreading may be characterised via suitable finite collections of weakly mean sensitive tuples. However, due to the fact that the spreading must occur along sets of full density, it turns out that this cannot be achieved by looking at single tuples. Instead, finite collections of suitable tuples have to be considered.

Recall that we call a tds (X,G) *strongly (ε,m)-spreading* for some $\varepsilon>0$ and $m\in\mathbb{N}_2$ if

$$\underline{\mathrm{FD}}\left(\{g\in G \mid \mathrm{diam}_m(gU)\geqslant\varepsilon\}\right) \;=\; 1$$

holds for any open subset $U\subseteq X$. Note that (X,G) is *strongly m-spreading* if it is strongly (ε,m)-spreading for some $\varepsilon>0$. Moreover, recall that we defined

$$D_m(x) \;=\; \min_{1\leq i<j\leq m} d(x_i,x_j)$$

and

$$X_\varepsilon^m \;=\; \{x\in X^m \mid D_m(x)\geq\varepsilon\}\,.$$

Now strong m-spreading can be characterised in the following way.

Lemma 5.4. *If (X,G) is strongly (ε,m)-spreading, then for all $\delta>0$ there exists $N=N(\delta)\in\mathbb{N}$ and $x^{(1)},\ldots,x^{(N)}\in X_\varepsilon^m$ such that*

$$\underline{\mathrm{FD}}\left(\left\{g\in G \;\middle|\; \exists k\in\{1,\ldots,N\}: gU\cap B\left(x_i^{(k)},\delta\right)\neq\emptyset \textit{ for } i=1,\ldots,m\right\}\right). \;=\; 1\,. \tag{5.4}$$

holds for all nonempty open sets $U\subseteq X$.

Proof. Note that if a set $V\subseteq X$ satisfies $\mathrm{diam}_m(V)>\varepsilon$, then V^m intersects X_ε^m. Hence, as (X,G) is strongly (ε,m)-spreading, we have that

$$\underline{\mathrm{FD}}\left(\{g\in G \mid (gU)^m\cap X^{m,\varepsilon}\neq\emptyset\}\right) \;=\; 1\,. \tag{5.5}$$

Now fix $\delta>0$. Due to compactness, there exists $N\in\mathbb{N}$ and $x^{(1)},\ldots,x^{(N)}\in X_\varepsilon^m$ such that the product neighborhoods

$$U\left(x^{(k)},\delta\right) \;=\; \bigotimes_{k=1}^m B\left(x_i^{(k)},\delta\right)$$

with $k=1,\ldots,N$ cover X_ε^m. As a consequence, (5.5) implies (5.4). ☐

Conversely, we have

Lemma 5.5. *Let (X,G) be a tds and $\varepsilon>0$ and $\delta\in(0,\frac{\varepsilon}{2})$. If there exists $N=N(\delta)\in\mathbb{N}$ and $x^{(1)},\ldots,x^{(N)}\in X_\varepsilon^m$ such that*

$$\underline{\mathrm{FD}}\left(\left\{g\in G \;\middle|\; \exists k\in\{1,\ldots,N\}: gU\cap B\left(x_i^{(k)},\delta\right)\neq\emptyset \textit{ for } i=1,\ldots,m\right\}\right) \;=\; 1$$

holds for all nonempty open sets $U\subseteq X$, then (X,G) is strongly $(\varepsilon-2\delta,m)$-spreading.

Proof. The statement follows directly from the observation that $x\in X_\varepsilon^m$ and $V\cap B(x_i,\delta)\neq\emptyset$ for $i=1,\ldots,m$ implies $\mathrm{diam}_m(V)>\varepsilon-2\delta$. ☐

Together, the previous two lemmas yield

Proposition 5.6. *Let (X,G) be a tds, $m\in\mathbb{N}_2$ and $\varepsilon_0>0$. Then the following are equivalent:*

(i) *(X,G) is strongly (ε,m)-spreading for all $\varepsilon<\varepsilon_0$;*
(ii) *for all $\varepsilon<\varepsilon_0$ and $\delta>0$ there exist $N=N(\varepsilon,\delta)$ and $x^{(1)},\ldots,x^{(N)}\in X_\varepsilon^m$ such that*

$$\underline{\mathrm{FD}}\left(\left\{g\in G \;\middle|\; \exists k\in\{1,\ldots,N\}: gU\cap B\left(x_i^{(k)},\delta\right)\neq\emptyset \textit{ for } i=1,\ldots,m\right\}\right) \;=\; 1$$

holds for all nonempty open sets $U\subseteq X$.

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