The Interplay between Additive and Multiplicative Central Sets Theorems

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Abstract

The concept of central sets, introduced by Furstenberg through the framework of topological dynamics, has played a pivotal role in combinatorial number theory. Furstenberg's Central Sets Theorem highlighted their rich combinatorial structure. Later in [Fund. Math., 199 (2008), 155175.], De, Hindman, and Strauss strengthen this theorem using the algebraic framework of the Stone–Čech compactification. In this article, we establish a unified version of the Central Sets Theorem that simultaneously captures both additive and multiplicative structures.

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1 Introduction

Let X be a compact Hausdorff space and $T : X \to X$ a continuous self-map. The pair (X, T) is what we call a **dynamical system**. Given an open set $U \subseteq X$ and a point $x \in X$, the set

$$R(x, U) = \{n \in \mathbb{N} \mid T^n x \in U\}$$

is known as the **return times set** of x to U.

Although this definition is presented in the context of a $\mathbb{Z}$-action, it naturally extends to more general dynamical systems that arise from actions of arbitrary semigroups.

In his work [3, Proposition 8.21], Furstenberg introduced the concept of central sets via return times and proved the celebrated Central Sets Theorem, which has since become a cornerstone in

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topological dynamics and combinatorics. Later, following [1, 9], central sets can also be characterized as members of minimal idempotent ultrafilters in the Stone–Čech compactification of a discrete semigroup. More explicitly, if $(S, \cdot)$ is any discrete semigroup and $(\beta S, \cdot)$ is the Stone–Čech compactification of S, then a set $A \subseteq S$ is said to be **central** if there exists a minimal idempotent ultrafilter p such that $A \in p$.

Throughout the paper, $\mathbb{N}$ denotes the set of positive integers. For a nonempty set X, we write $\mathcal{P}_f(X)$ for the family of all nonempty finite subsets of X, and $X^{\mathbb{N}}$ for the set of sequences in X.

We now state the Central Sets Theorem.

Theorem 1.1 (**Central Sets Theorem**). *Let $l \in \mathbb{N}$, and let $A \subseteq \mathbb{N}$ be a central set. For each $i \in \{1, 2, \ldots, l\}$, let $\langle x_{i,m} \rangle_{m=1}^{\infty}$ be a sequence in $\mathbb{N}$. Then there exist a sequence $\langle b_m \rangle_{m=1}^{\infty}$ in $\mathbb{N}$ and a sequence $\langle K_m \rangle_{m=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that:*

1. *For all m, $\max K_m < \min K_{m+1}$.*
2. *For every $i \in \{1, 2, \ldots, l\}$ and any $H \in \mathcal{P}_f(\mathbb{N})$,*

$$\sum_{m \in H} \left(b_m + \sum_{t \in K_m} x_{i,t} \right) \in A.$$

Using the Stone–Čech compactification of discrete semigroups, D. De, N. Hindman, and D. Strauss obtained in [2] a strengthened form of the Central Sets Theorem, now known as the **Stronger Central Sets Theorem**.

Theorem 1.2 (**Stronger Central Sets Theorem**). *Let $(S, +)$ be a commutative semigroup, $\tau = {}^{\mathbb{N}}S$, and let $C \subseteq S$ be a central set. Then there exist functions*

$$\alpha : \mathcal{P}_f(\tau) \to S \quad \textit{and} \quad H : \mathcal{P}_f(\tau) \to \mathcal{P}_f(\mathbb{N})$$

such that:

1. *For all $F, G \in \mathcal{P}_f(\tau)$ with $F \subsetneq G$,*

$$\max H(F) < \min H(G).$$

2. *For any $m \in \mathbb{N}$, and any sequence*

$$G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_m \quad \textit{in } \mathcal{P}_f(\tau),$$

with $f_i \in G_i$ for each $i = 1, 2, \ldots, m$, we have

$$\sum_{i=1}^{m} \left(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t) \right) \in C.$$

For a noncommutative version of the above theorem, we refer to the article [8].

In this article, our primary goal is to prove a version of the Central Sets Theorem, which unifies both the additive and multiplicative Central Sets Theorems, that means it unifies the Central Sets Theorem on both $(\mathbb{N},+)$ and $(\mathbb{N},\cdot)$. The core idea behind this result arises from the following observation: for any finite coloring of $\mathbb{N}$, there exists a color class that is both an additive and a multiplicative central set. Consequently, it satisfies both versions of the Central Sets Theorem. This naturally leads to the following question.

How do these two Central Sets Theorems interact with each other?

In this paper, we explore the answer to this question, where we prove a mixture of both additive and multiplicative Central Sets Theorem, with a particular focus on the function $H : \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N}) \to \mathcal{P}_f(\mathbb{N})$ in Theorem 1.2. Note that our function H is the same for both semigroups $(\mathbb{N},+)$ and $(\mathbb{N},\cdot)$.

For any semigroup $(S,\cdot)$, let $E(\beta S,\cdot)$ be the collection of all idempotent ultrafilters. The following lemma is standard.

Lemma 1.3. *The set* $\overline{E\left(K\left(\beta\mathbb{N},+\right)\right)} \cap E\left(K\left(\beta\mathbb{N},\cdot\right)\right)$ *is nonempty.*

The following theorem is our main theorem.

Theorem 1.4 (**Unified Central Sets Theorem**). *Let* $p \in \overline{E\left(K\left(\beta\mathbb{N},+\right)\right)} \cap E\left(K\left(\beta\mathbb{N},\cdot\right)\right)$ *and* $A \subseteq \mathbb{N}$ *be such that* $A \in p$, *that is* A *is both an additive and a multiplicative central set.*

Then there exist a color class $A \subseteq \mathbb{N}$ *and three functions*

$$\alpha,\beta : \mathcal{P}_f\left({}^{\mathbb{N}}\mathbb{N}\right) \to \mathbb{N} \quad \text{and} \quad H : \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N}) \to \mathcal{P}_f(\mathbb{N})$$

such that:

1. *If* $F, G \in \mathcal{P}_f\left({}^{\mathbb{N}}\mathbb{N}\right)$ *and* $F \subsetneq G$, *then*
$$\max H(F) < \min H(G).$$
2. *For any* $m \in \mathbb{N}$ *and* $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_m$ *in* $\mathcal{P}_f\left({}^{\mathbb{N}}\mathbb{N}\right)$, *and for each* $i = 1, 2, \ldots, m$ *with* $f_i \in G_i$, *we have:*
   - (a) *for any* $K \subseteq \{1,\ldots,m\}$
   $$\sum_{i\in K}\left(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t)\right) \in A.$$
   - (b) *for any* $L \subseteq \{1,\ldots,m\}$
   $$\prod_{j\in L}\left(\beta(G_i)\cdot \prod_{t\in H(G_i)} f_i(t)\right) \in A.$$
   - (c) *For all* $N \leqslant m$, $K \subseteq \{1,\ldots N\}$ *and* $L \subseteq \{N+1,\ldots m\}$
   $$\left(\sum_{i\in K}\left(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t)\right)\right)\cdot\left(\prod_{j\in L}\left(\beta(G_i)\cdot \prod_{t\in H(G_i)} f_i(t)\right)\right) \in A.$$

2 Preliminaries on Partial Semigroups

Here we recall some basic facts about partial semigroups. For details, we refer to the article [5].

Definition 2.1. A *partial semigroup* is a pair $(S, \cdot)$, where "$\cdot$" is a partially defined binary operation on S; that is, it maps a subset of $S \times S$ into S. It satisfies the following associativity condition:

for all $a, b, c \in S$, if either $(a \cdot b) \cdot c$ or $a \cdot (b \cdot c)$ is defined, then so is the other, and they are equal.

Example 1.

(1) Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in a semigroup $(S, \cdot)$, and let

$$FP\left(\langle x_n \rangle\right) = \left\{ \prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N}) \right\},$$

where the product is taken in increasing order of indices. In general, this set is not closed under the semigroup operation. However, define

$$\left(\prod_{n \in F} x_n \right) \cdot \left(\prod_{n \in G} x_n \right) = \begin{cases} \prod_{n \in F \cup G} x_n & \text{if } \max F < \min G, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then $(T, \cdot)$, where $T = FP(\langle x_n \rangle)$, is a partial semigroup.

The study of partial semigroups plays an important role in Ramsey theory. In this paper, we focus on a special class of partial semigroups known as *adequate partial semigroups*, which are particularly useful for combinatorial applications.

Definition 2.2 (Adequate Partial Semigroup). Let $(S, \cdot)$ be a partial semigroup.

(1) For $s \in S$, define $\phi(s) = \{t \in S : s \cdot t \text{ is defined}\}$.

(2) For $H \in \mathcal{P}_f(S)$, define $\sigma(H) = \bigcap_{s \in H} \phi(s)$.

(3) The partial semigroup $(S, \cdot)$ is said to be *adequate* if $\sigma(H) \neq \varnothing$ for all finite $H \subseteq S$.

In Example 1, one can verify that unlike $(\mathcal{R}, \cdot)$, the partial semigroup $(T, \cdot)$ is *adequate.*

In the case of $(\mathcal{R}, \cdot)$, observe that for any finite subset $H \subseteq \mathcal{R}$, we have $\sigma(H) \neq \varnothing$ if and only if all matrices in H have the same number of columns. That is, if we define

$$\mathcal{H} = \{A \in \mathcal{R} : A \text{ is a matrix with } r \text{ columns for some fixed } r \in \mathbb{N}\},$$

then $\sigma(H) \neq \varnothing$ if and only if $H \subseteq \mathcal{H}$ for some fixed r. Therefore, $(\mathcal{R}, \cdot)$ is not adequate in general.

2.0.1 Algebra of the Stone-Čech Compactification of Discrete Partial Semigroups

Let $(S, \cdot)$ be a partial semigroup, and let βS denote the set of ultrafilters on S. Then $(\beta S, \cdot)$ is also a partial semigroup.

One of the key advantages of working with *adequate partial semigroups* is that, unlike general partial semigroups, we can identify a genuine semigroup structure within βS.

Definition 2.3. Given a partial semigroup $(S, \cdot)$, define:

$$\delta S = \bigcap_{x \in S} \overline{\phi(x)} = \bigcap_{H \in \mathcal{P}_f(S)} \overline{\sigma(H)}.$$

Clearly $\delta S \subseteq \beta S$. Remarkably, this set δS, equipped with a suitably defined operation, forms a semigroup.

For $x \in S$ and $A \subseteq S$, define:

$$x^{-1}A = \{y \in \phi(x) : x \cdot y \in A\}.$$

The following lemma describes the algebraic structure of adequate partial semigroups within the StoneČech compactification.

Lemma 2.4. [5, Lemma 2.4] *Let S be an adequate partial semigroup.*

(a) Let $x \in S$, $q \in \overline{\phi(x)}$, and $A \subseteq S$. Then

$$A \in x \cdot q \iff x^{-1}A \in q.$$

(b) Let $p \in \beta S$, $q \in \delta S$, and $A \subseteq S$. Then

$$A \in p \cdot q \iff \{x \in S : x^{-1}A \in q\} \in p.$$

The following theorem guarantees the existence of idempotents in partial semigroups.

Theorem 2.5. [5, Theorem 2.6] *Let S be an adequate partial semigroup. Then, with the operation described above, δS is a compact right topological semigroup and hence contains idempotents.*

3 Proof of Theorem 1.4

Let S be a semigroup. For $m \in \mathbb{N}$, define

$$\mathcal{I}_m = \left\{ (H_1, H_2, \ldots, H_m) : \begin{array}{l} \text{each } H_j \in \mathcal{P}_f(\mathbb{N}) \text{ and} \\ \text{for all } j \in \{1, 2, \ldots, m-1\},\ \max H_j < \max H_{j+1} \end{array} \right\}.$$

Given $m \in \mathbb{N}$, $a \in S^{m+1}$, $H \in \mathcal{I}_m$, and $f \in {}^{\mathbb{N}}S$, define

$$x(m, a, H, f) = \left(\prod_{j=1}^{m} \left(a(j) \cdot \prod_{t \in H_j} f(t) \right) \right) \cdot a(m+1).$$

Let $\mathcal{I} = \bigcup_{m=1}^{\infty} \mathcal{I}_m$, and define a partial binary operation $*$ on $\mathcal{I}$ by

$$(H_1, \ldots, H_m) * (K_1, \ldots, K_n) = (H_1, \ldots, H_m, K_1, \ldots, K_n),$$

whenever $\max H_m < \min K_1$; otherwise, the operation is undefined.

It follows from [6] that $(\mathcal{I}, *)$ is an adequate partial semigroup, and

$$\delta\mathcal{I} = \bigcap_{n=1}^{\infty} \overline{\{H \in \mathcal{I} : \min H_1 > n\}}^{\mathcal{I}}.$$

We need the following lemma to prove our main result.

Lemma 3.1. *[6, Lemmas 2.10] Let $(S, \cdot)$ be a partial semigroup, $A \subseteq S$ a piecewise syndetic set, and $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$. Define*

$$\mathcal{B}(A, F, \cdot) = \left\{(H_1, \ldots, H_m) \in \mathcal{I} : \exists a \in S^{m+1} \text{ such that } \forall f \in F,\ x(m, a, H, f) \in A\right\}.$$

Then for every $p \in E(\delta\mathcal{I})$, we have $\mathcal{B}(A, F, \cdot) \in p$.

Now we are in the position to prove our main result.

Proof of Theorem 1.4: Let

$$p \in \overline{E\left(K\left(\beta\mathbb{N}, +\right)\right)} \cap E\left(K\left(\beta\mathbb{N}, \cdot\right)\right),$$

and let $A \in p$. Then A is both additively and multiplicatively central.

Define

$$A^\star(p) = \{x \in A : x^{-1}A \in p\}.$$

As p is fixed, we will write $A^\star$ instead of $A^\star(p)$. Since $p \cdot p = p$, it follows that $A^\star \in p$. Moreover, by [7, Lemma 4.14], if $x \in A^\star$, then

$$x^{-1}A^\star \in p.$$

Now, by the choice of p, we have $A^\star \in q$ for some additive minimal idempotent $q \in E(K(\beta\mathbb{N}, +))$.

Define

$$A^{\star\star}(q) = \{x \in A^\star : -x + A^\star \in q\}.$$

Then $A^{\star\star}(q) \in q$. Now again, as q is fixed, we will write $A^\star = A^{\star\star}(q) \in q$. Furthermore, for any $x \in A^{\star\star}$, it follows that

$$-x + A^{\star\star} \in q.$$

We define $\alpha(F), \beta(F) \in \mathbb{N}$ and $H(F) \in \mathcal{P}_f(\mathbb{N})$ for $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ by induction on $|F|$, satisfying the following inductive hypothesis:

(1) For $F, G \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ with $\varnothing \neq G \subsetneq F$, we have

$$\max H(G) < \min H(F).$$

(2) Whenever $n \in \mathbb{N}$, $G_1, G_2, \ldots, G_n \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ with

$$G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n = F,$$

and for each $i \in \{1, 2, \ldots, n\}$, $f_i \in G_i$, then the following hold:

(a) for any $K \subseteq \{1, \ldots, m\}$

$$\sum_{i \in K} \left(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t) \right) \in A^{\star\star} \subset A^{\star}.$$

(b) for any $L \subseteq \{1, \ldots, m\}$

$$\prod_{j \in L} \left(\beta(G_j) \cdot \prod_{t \in H(G_j)} f_j(t) \right) \in A^{\star}.$$

(c) For all $N \leqslant n$, $K \subseteq \{1, \ldots N\}$ and $L \subseteq \{N+1, \ldots n\}$,

$$\left(\sum_{i \in K} \left(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t) \right) \right) \cdot \left(\prod_{j \in L} \left(\beta(G_j) \cdot \prod_{t \in H(G_j)} f_j(t) \right) \right) \in A^{\star}.$$

Let $F = \{f\}$. Since $A^{\star\star}$ is an additively piecewise syndetic set, it follows from Lemma 3.1 that there exist $a \in \mathbb{N}$ and $L_0 \in \mathcal{P}_f(\mathbb{N})$ such that

$$a + \sum_{t \in L_0} f(t) \in A^{\star\star}.$$

Define

$$B = A^{\star\star} \cap \left(- \left(a + \sum_{t \in L_0} f(t) \right) + A^{\star\star} \right) \in q \quad \text{and} \quad C = A^{\star} \cap \left(a + \sum_{t \in L_0} f(t) \right)^{-1} A^{\star} \in p.$$

Using Lemma 3.1 again, and noting that B and C are additively and multiplicatively piecewise syndetic sets respectively, for every $p \in E(\delta\mathcal{I})$, we have $\mathcal{B}(B, \{f\}, +) \in p$ and $\mathcal{B}(C, \{f\}, \cdot) \in p$. Hence for every $p \in E(\delta\mathcal{I})$, we have

$$\mathcal{B}(B, \{f\}, +) \cap \mathcal{B}(C, \{f\}, \cdot) \in p.$$

In particular, there exist $a_1, b_1 \in \mathbb{N}$ and $L_1 \in \mathcal{P}_f(\mathbb{N})$ such that

$$a_1 + \sum_{t \in L_1} f(t) \in B \quad \text{and} \quad b_1 \cdot \prod_{t \in L_1} f(t) \in C.$$

Consequently, we obtain:

- $a_0 + \sum_{t\in L_0} f(t),\ a_1 + \sum_{t\in L_1} f(t) \in A^\star$,
- $b_1 \cdot \prod_{t\in L_1} f(t) \in A^\star$,
- $\big(a_0 + \sum_{t\in L_0} f(t)\big) \cdot \big(b_1 \cdot \prod_{t\in L_1} f(t)\big) \in A^\star$.

Define $\alpha(\{f\}) = a_1$ and $H(\{f\}) = L_1$. Then both inductive hypotheses are satisfied in this base case.

Now assume $|F| > 1$, and suppose that for all proper subsets $G \subsetneq F$, the values $\alpha(G) \in \mathbb{N}$ and $H(G) \in \mathcal{P}_f(\mathbb{N})$ have already been defined satisfying the inductive conditions.

Let $K = \bigcup \{H(G) : \varnothing \neq G \subsetneq F\}$, and set $m = \max K$. Define the sets:

$$M_1 = \left\{ \sum_{i\in K} \left(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t) \right) : K \subseteq \{1,\dots,n\}\ \textit{and}\ \begin{array}{l} n\in\mathbb{N},\ G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n \subsetneq F, \\ \langle f_i \rangle_{i=1}^n \in \times_{i=1}^n G_i \end{array} \right\},$$

$$M_2 = \left\{ \prod_{j\in L} \left(\beta(G_j) \cdot \prod_{t\in H(G_j)} f_j(t) \right) : L \subseteq \{1,\dots,n\}\ \textit{and}\ \begin{array}{l} n\in\mathbb{N},\ G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n \subsetneq F, \\ \langle f_j \rangle_{j=1}^n \in \times_{i=1}^n G_i \end{array} \right\}.$$

and

$$\begin{aligned} M_3 = \Bigg\{ \left(\sum_{i\in K} \left(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t) \right) \right) \cdot \left(\prod_{j\in L} \left(\beta(G_j) \cdot \prod_{t\in H(G_j)} f_j(t) \right) \right) : \\ n, N \in \mathbb{N} \text{ with } N \leqslant n, K \subseteq [1,N], L \subseteq [N+1,n] \\ G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n \subsetneq F,\ \langle f_i \rangle_{i=1}^n \in \times_{i=1}^n G_i \Bigg\}. \end{aligned}$$

Let

$$B_1 = A^{\star\star} \cap \bigcap_{x\in M_1} \left(-x + A^{\star\star}\right).$$

and

$$C_1 = A^\star \cap \bigcap_{x\in M_1} \left(x^{-1}A^\star\right) \bigcap_{x\in M_2} \left(x^{-1}A^\star\right) \bigcap_{x\in M_3} \left(x^{-1}A^\star\right).$$

By hypothesis (2), $M_1 \subset A^{\star\star} \subset A^\star$ and $M_2 \subset A^\star$, which, together with the fact that the sets M_i for $i = 1,2,3$ are finite, implies that

$$B_1 \in q \quad \text{and} \quad C_1 \in p.$$

Thus, Lemma 3.1 similarly guarantees that there exist $a, b \in \mathbb{N}$ and $L \in \mathcal{P}_f(\mathbb{N})$ such that $\min L > m$, and for all $f \in F$,

$$a + \sum_{t\in L} f(t) \in B_1 \quad \text{and} \quad b \cdot \prod_{t\in L} f(t) \in C_1.$$

Let $\alpha(F) = a$, $\beta(F) = b$, and $H(F) = L$. Since $\min L > m$, the first inductive hypothesis is satisfied. We now verify hypothesis (2). If $n = 1$, then we have

$$a + \sum_{t\in L} f(t) \in B_1 \subset A^{\star\star} \subseteq A^{\star} \quad \text{and} \quad b \cdot \prod_{t\in L} f(t) \in C_1 \subseteq A^{\star},$$

which shows that hypothesis (2) holds in this case.

And if $n > 1$, and $K, L \subseteq [1, n]$ let

$$x_K = \sum_{i\in K} \left(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t) \right), \quad y_L = \prod_{j\in L} \left(\beta(G_j) \cdot \prod_{t\in H(G_j)} f_j(t) \right).$$

And for any $N \leqslant n$, letting $K \subseteq [1, N]$ and $L \subseteq [N+1, n]$ let

$$z_{K,L} = \left(\sum_{i\in K} \left(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t) \right) \right) \cdot \left(\prod_{j\in L} \left(\beta(G_j) \cdot \prod_{t\in H(G_j)} f_j(t) \right) \right).$$

Then $x_K \in M_1$, $y_L \in M_2$ and $z_{K,L} \in M_3$, and hence for every such $K, L \subseteq [1, n]$, we have

- $a + \sum_{t\in L} f(t) \in B_1 \subseteq -x_k + A^{\star\star} \subseteq -x_k + A^{\star}$;
- $b \cdot \prod_{t\in L} f(t) \in C_1 \subseteq y_l^{-1} A^{\star}$;
- For $N \leqslant n$, $b \cdot \prod_{t\in L} f(t) \in C_1 \subset z_{K,L}^{-1} A^{\star}$

for all $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n = F$.

Therefore, hypothesis (2) holds for all $n \in \mathbb{N}$. This completes the inductive proof of the central set theorem. $\square$

3.1 Applications of Theorem 1.4

As we have already mentioned before Lemma 1.3 that main significance of our Theorem 1.4 is that the function $H : \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N}) \to \mathcal{P}_f(\mathbb{N})$ is same for both the semigroups $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$. Here we give a few applications. The classical van der Waerden's theorem says that additively piecewise syndetic sets contain arithmetic progressions of arbitrary length. The same is true if one replaces the additive piecewise syndetic sets by multiplicative piecewise syndetic sets, and arithmetic progressions by geometric progressions. The following corollary is the first application of our Theorem 1.4, which gives a mixture of these two theorems.

Corollary 3.2. *If $\mathbb{N}$ is finitely coloured, then for every $d \in \mathbb{N}$ there exist $a_1, a_2, b_1, b_2, n_1, n_2 \in \mathbb{N}$ such that the set*

$$\left\{ \begin{array}{ccc} a_1 + n_1 d, & a_2 + n_2 d, & (a_1 + a_2) + (n_1 + n_2) d \\ b_1 d^{n_1}, & b_2 d^{n_2}, & b_1 b_2 d^{n_1 + n_2} \\ (a_1 + n_1 d) \cdot b_2 d^{n_2} & & \end{array} \right\}.$$

is monochromatic.

Proof. The proof of this corollary is immediate if one chooses the constant sequence $G_1 = \{f : \mathbb{N} \to \mathbb{N} : f(n) = d$ for all $n \in \mathbb{N}\}$, in Theorem 1.4. □

If we choose the set of the same sequence G_1 as in the above Theorem, then Theorem 1.4 implies the following corollary.

Corollary 3.3. *If $\mathbb{N}$ is finitely colored, then for any given $N \in \mathbb{N}$, there exist two sequences $\langle x_n \rangle_{n=1}^N$ and $\langle y_n \rangle_{n=1}^N$ such that the following sets are simultaneously monochromatic.*

$$FS(\langle x_n \rangle_{n=1}^N) \cup FP(\langle y_n \rangle_{n=1}^N) \cup \left\{ \left(\sum_{i \in H} x_i \right) \cdot \left(\prod_{j \in K} y_j \right) : \max H < \min K \right\}.$$

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