METRIC MEAN DIMENSION VIA PREIMAGE STRUCTURES

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Abstract. The preimage entropy provides a quantitative estimate of how "invertible" a system is. Once there are several examples where the preimage entropy is infinite, it cannot provide more information. Thus, we introduce the concept of preimage metric mean dimension, and study many properties of it. Meanwhile, we provide many examples to compute their preimage mean metric dimension.

1. Introduction

Let X be a compact topological space, and $f : X \to X$ be a continuous map. One of the most important notions in Dynamical Systems is that of *topological entropy*, denoted by $h_{\mathrm{top}}(f)$ the topological entropy of f on the space X. It is a topological invariant and, roughly speaking, measures how chaotic a system is, and may be thought as a quantity that measures the rate disperses points in the future by the action of f on the space X. In particular, it is an effective tool to decide whether two systems are conjugated or not.

When the map f under consideration is a homeomorphism, then extending this procedure into the past instead of the future results is the entropy $h_{\mathrm{top}}(f^{-1})$ of the inverse mapping, which coincides with $h_{\mathrm{top}}(f)$. However, when the map is not invertible, different ways of "extending the procedure into the past" lead to several new entropy-like invariants for noninvertible maps. In [13], Hurley introduced several other entropy-like invariants for noninvertible maps, mentioned sometimes as topological entropy via preimage structures: the *preimage branch entropy*, denoted by $h_{\mathrm{b}}(f)$, distinguishing points according to the branches of the inverse map; and two *pointwise preimage entropies* $h_{\mathrm{p}}(f)$ and $h_{\mathrm{m}}(f)$. In [7], Fiebig, Fiebig and Nitecki, introduced the notion of *preimage relation entropy*, denoted by $h_{\mathrm{r}}(f)$ and exploited these four concepts, showing that they behave in ways that are often unexpected in light of the well understood behavior of topological entropy $h_{\mathrm{top}}(f)$. As in the case of topological entropy, some authors turn out to the problem of defining a measure-theoretical version of the preimage entropies. In [6], Cheng and Newhouse introduced another notion of preimage entropy, denoted by $h_{\mathrm{pre}}(f)$, and associated to such quantity a notion of preimage metric entropy was defined and a partial variational principle was obtained. More recently, Wu and Zhu [27, 28] defined the concept of preimage metric entropy related to $h_{\mathrm{m}}(f)$ and proved that for maps with uniform separation of preimages property it is possible to guarantee a variational principle relating such quantities.

In the case of topological entropy, there are plenty of systems with infinite topological entropy (for instance, they form a C^0-generic set in the space of homeomorphisms of a compact manifold [23] with dimension greater than one) and thus, in this context, the entropy is not useful anymore. Therefore, to study these types of

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2020 *Mathematics Subject Classification.* Primary: 37A35, 37B40, 37D35; Secondary: 37A05, 37B05, systems, new dynamical quantities are required and an example of such a quantity is the *metric mean dimension*.

The notion of metric mean dimension was introduced by Lindenstrauss and Weiss in [16] as metric-dependent analog of the *mean dimension*, another topological invariant of high-complexity maps which was introduced by Gromov [8]. The definition of metric mean dimension is a fusion of the definitions of topological entropy and Minkowski dimension and has several applications, like in the study of embedding problems [12], and the metric mean dimension presents an upper bound to it. But more than that, the metric mean dimension turned out to be useful in several contexts like in the study of compression [9, 10]. One of the main goals of this quantity was to bound from above the mean dimension.

Inspired by these works, we extend the notion of *metric mean dimension via preimage structures*. Examples of maps with infinite preimage entropy may be easily constructed and because of this we believe that such a notion represents an interesting contribution to the study of continuous dynamics. We present several examples, and prove that, in certain setting, the preimage metric mean dimension can be computed via a variational principle.

This manuscript is organized as follows. In Section 2, we introduce main definitions and present our main results and in the remaining sections we prove our main results. In the last section, we will provide some examples.

2. Definitions and Statements

Let (X,d) be a compact metric space and $f\colon X\to X$ be a continuous map. Given $n\in\mathbb{N}$, we define the dynamical metric $d_n\colon X\times X\to[0,\infty)$ by

$$d_n(x,z)=\max\Big\{d(x,z),\, d(f(x),f(z)),\,\dots,\, d(f^{n-1}(x),f^{n-1}(z))\Big\}.$$

It is easy to see that d_n is indeed a metric and, moreover, generates the same topology as d. Furthermore, given $\varepsilon>0$, $n\in\mathbb{N}$ and a point $x\in X$, we define the open (n,ε)-ball around x by

$$B_n(x,\varepsilon)=\{y\in X: d_n(x,y)<\varepsilon\}.$$

We say a set $E\subset X$ is (n,ε)*-separated* by f if $d_n(x,z)>\varepsilon$ for every $x\neq z\in E$. Denote by $s(f,n,\varepsilon)$ the maximal cardinality of all (n,ε)–separated subsets of X by f. We say $R\subset X$ is a (n,ε)*-spanning set* if for any $x\in X$, there exists $z\in R$ such that $d_n(x,z)<\varepsilon$. Denote by $r(f,n,\varepsilon)$ the minimal cardinality of all (n,ε)-spanning subsets of X by f. Due to the compactness of X, $s(f,n,\varepsilon)$ and $r(f,n,\varepsilon)$ are finite for any $n\in\mathbb{N}$ and $\varepsilon>0$.

2.1. **The metric mean dimension.**

The *upper metric mean dimension* of f with respect to d is given by

$$\overline{\mathrm{mdim}}_{\mathrm{M}}\Big(X,f,d\Big)=\limsup_{\varepsilon\to 0}\frac{h(f,\varepsilon)}{|\log\varepsilon|},$$

where

$$h(f,\varepsilon)=\limsup_{n\to\infty}\frac{1}{n}\log s(f,n,\varepsilon).$$

Similarly, the *lower metric mean dimension* of f with respect to d is given by

$$\underline{\mathrm{mdim}}_{\mathrm{M}}\Big(X,f,d\Big)=\liminf_{\varepsilon\to 0}\frac{h(f,\varepsilon)}{|\log\varepsilon|}.$$

In the case when $\underline{\mathrm{mdim}}_{\mathrm{M}}\Big(X,f,d\Big)=\overline{\mathrm{mdim}}_{\mathrm{M}}\Big(X,f,d\Big)$ this common value is called the *metric mean dimension* of f with respect to d and is denoted simply by $\mathrm{mdim}_{\mathrm{M}}\Big(X,f,d\Big)$.

Recall that the *topological entropy* of the map f is given by

$$h_{\mathrm{top}}(f)=\lim_{\varepsilon\to 0}h(f,\varepsilon).$$

Consequently, $\overline{\mathrm{mdim}}_{\mathrm{M}}\Big(X,f,d\Big)=\underline{\mathrm{mdim}}_{\mathrm{M}}\Big(X,f,d\Big)=0$ whenever the topological entropy of f is finite. In particular, the metric mean dimension is a suitable quantity to study systems with infinite topological entropy. For more on these quantities see [15, 16, 20] and references therein.

2.2. **Variational principle for metric mean dimension.** Let $\mathcal{B}$ be the Borel σ-algebra on X. Denote by $\mathcal{M}_f(X)$ the set of all f-invariant Borel probability measures on X, and by $\mathcal{M}_f^e(X)$ the set of all ergodic Borel probability measures on X. Given $\mu\in\mathcal{M}_f(X)$, we say that $\xi=\{C_1,\dots,C_k\}$ is a finite measurable partition of X if $\xi\subset\mathcal{B}$, $\mu\left(X\setminus\cup_{i=1}^k C_i\right)=0$ and $\mu\left(C_i\cap C_j\right)=0$ for every $i\neq j$. The set of all finite Borel partitions of X is denoted by $\mathcal{P}(X)$. Given $\varepsilon>0$, let $\mathcal{P}_\varepsilon(X)=\{\xi\in\mathcal{P}(X):|\xi|<\varepsilon\}$, where $|\xi|=\max\{\mathrm{diam}(A):A\in\xi\}$.

The *entropy* of $\xi\in\mathcal{P}(X)$ with respect to μ is given by

$$H_\mu(\xi)=-\sum_{C\in\xi}\mu(C)\log\mu(C).$$

Given $\xi,\zeta\in\mathcal{P}(X)$, denote $\xi\vee\zeta=\{A\cap B:A\in\xi,B\in\zeta\}\in\mathcal{P}(X)$. Let $\xi^n=\bigvee_{j=0}^{n-1}f^{-j}\xi$ for any $\xi\in\mathcal{P}(X)$ and $n\in\mathbb{N}$. Then, the *metric entropy of* ξ *with respect to* (f,μ) is given by

$$h_\mu(f,\xi)=\lim_{n\to+\infty}\frac{1}{n}H_\mu(\xi^n).$$

The *metric entropy with respect to* (f,μ) is given by

$$h_\mu(f)=\sup_{\xi\in\mathcal{P}(X)}h_\mu(f,\xi)=\lim_{\varepsilon\to 0}\inf_{\xi\in\mathcal{P}_\varepsilon(X)}h_\mu(f,\xi).$$

Following this idea, the metric mean dimension in measure-theoretical sense is defined by

$$\mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M}}\left(f,d\right)=\limsup_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\sup_{\mu\in\mathcal{M}_f(X)}\inf_{\xi\in\mathcal{P}_\varepsilon(X)}h_\mu(f,\xi).\tag{1}$$

Gutman [11] proved that the metric mean dimension of a continuous endomorphism on a compact metric space satisfies the following variational principle

$$\overline{\mathrm{mdim}}_{\mathrm{M}}\Big(X,f,d\Big)=\mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M}}\Big(X,f,d\Big).\tag{2}$$

2.3. **Topological preimage entropy and preimage metric mean dimension.**

2.3.1. *Topological preimage entropy.* Fix $x\in X$. Given $n\in\mathbb{N}$ and $\varepsilon>0$, let us denote by $s(n,\varepsilon,f^{-n}(x))$ the maximal cardinality of all (n,ε)–separated subsets of $f^{-n}(x)$ with respect to f, and by $r(n,\varepsilon,f^{-n}(x))$ the minimal cardinality of all (n,ε)–spanning subsets of $f^{-n}(x)$ with respect to f.

The *topological preimage entropy* of f, defined by Hurley (see [13, 18] for more details) is given by

$$h_{\mathrm{m}}\left(f\right)=\lim_{\varepsilon\to 0}h_{\mathrm{m}}(f,\varepsilon),\tag{3}$$

where

$$h_{\mathrm{m}}(f,\varepsilon)=\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X}s(n,\varepsilon,f^{-n}(x)).$$

Another possible definition of preimage entropy, introduced by Cheng and Newhowse [6], is defined via

$$h_{\mathrm{pre}}(f) = \lim_{\varepsilon\to 0} h_{\mathrm{pre}}(f,\varepsilon),$$

where

$$h_{\mathrm{pre}}(f,\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X,\, k\geq n} s(n,\varepsilon,f^{-k}(x)).$$

Remark 2.1. It is a direct consequence of the previous definitions that

$$h_{\mathrm{m}}(f) \leq h_{\mathrm{pre}}(f) \leq h_{\mathrm{top}}(f).$$

2.3.2. *Preimage metric mean dimension.* Motivated by previous definitions of preimage entropy, we introduce two possible notions of *upper metric mean dimension via preimage structures* as follows:

$$\overline{\mathrm{mdim}}_{\mathrm{M,m}}\Big(X,f,d\Big) = \limsup_{\varepsilon\to 0} \frac{h_{\mathrm{m}}(f,\varepsilon)}{|\log\varepsilon|} \text{ and } \overline{\mathrm{mdim}}_{\mathrm{M,pre}}\Big(X,f,d\Big) = \limsup_{\varepsilon\to 0} \frac{h_{\mathrm{pre}}(f,\varepsilon)}{|\log\varepsilon|}.$$

Similarly, the notions of *lower metric mean dimension via preimage structures* as follows:

$$\underline{\mathrm{mdim}}_{\mathrm{M,m}}\Big(X,f,d\Big) = \liminf_{\varepsilon\to 0} \frac{h_{\mathrm{m}}(f,\varepsilon)}{|\log\varepsilon|} \text{ and } \underline{\mathrm{mdim}}_{\mathrm{M,pre}}\Big(X,f,d\Big) = \liminf_{\varepsilon\to 0} \frac{h_{\mathrm{pre}}(f,\varepsilon)}{|\log\varepsilon|}.$$

When $\underline{\mathrm{mdim}}_{\mathrm{M,m}}\Big(X,f,d\Big) = \overline{\mathrm{mdim}}_{\mathrm{M,m}}\Big(X,f,d\Big)$ this common value is denoted by $\mathrm{mdim}_{\mathrm{M,m}}\Big(X,f,d\Big)$. Similarly, we denote by $\mathrm{mdim}_{\mathrm{M,pre}}\Big(X,f,d\Big)$ the common value when $\underline{\mathrm{mdim}}_{\mathrm{M,pre}}\Big(X,f,d\Big) = \overline{\mathrm{mdim}}_{\mathrm{M,pre}}\Big(X,f,d\Big)$

Remark 2.2. The following can be obtained by definitions

$$\overline{\mathrm{mdim}}_{\mathrm{M,m}}\Big(X,f,d\Big) \leq \overline{\mathrm{mdim}}_{\mathrm{M,pre}}\Big(X,f,d\Big) \leq \overline{\mathrm{mdim}}_{\mathrm{M}}\Big(X,f,d\Big).$$

and

$$\underline{\mathrm{mdim}}_{\mathrm{M,m}}\Big(X,f,d\Big) \leq \underline{\mathrm{mdim}}_{\mathrm{M,pre}}\Big(X,f,d\Big) \leq \underline{\mathrm{mdim}}_{\mathrm{M}}\Big(X,f,d\Big).$$

2.4. Measure-theoretic preimage entropy and preimage metric mean dimension.

2.4.1. *Measure-theoretic preimage entropy.* For $\mu \in \mathcal{M}_f(X)$ and $\xi \in \mathcal{P}(X)$, the *measure-theoretic preimage entropy* [6] was defined as

$$h_{\mathrm{pre},\mu}(f) = \sup_{\xi\in\mathcal{P}(X)} h_{\mathrm{pre},\mu}(f,\xi),$$

where $h_{\mathrm{pre},\mu}(f,\xi) = \lim_{n\to\infty} \frac{1}{n} H(\xi^n|\mathcal{B}^-)$ and $\mathcal{B}^-$ is the infinite past σ-algebra $\bigcap_{n\geq 0} f^{-n}\mathcal{B}$ related to the Borel σ-algebra $\mathcal{B}$ and $H(\cdot|\cdot)$ is the standard conditional entropy (see for example [24]). Authors [6] proved that

$$\sup_{\mu\in\mathcal{M}_f(X)} h_{\mathrm{pre},\mu}(f) \leq h_{\mathrm{pre}}(f).$$

Recently, the *pointwise metric preimage entropy with respect to* ξ [28], was defined by

$$h_{\mathrm{m},\mu}(f,\xi) = \limsup_{n\to+\infty} \frac{1}{n} H_\mu(\xi^n|f^{-n}\mathcal{B}).$$

The pointwise metric preimage entropy was defined as

$$h_{\mathrm{m},\mu}(f) = \sup_{\xi\in\mathcal{P}(X)} h_{\mathrm{m},\mu}(f,\xi).$$

Authors [28] proved that

$$\sup_{\mu\in\mathcal{M}_f(X)} h_{\mathrm{m},\mu}(f) \leq h_{\mathrm{m}}(f).$$

More recently, Wu and Zhu [27, Proposition 3.1] proved the two preimage entropy with respect to invariant probability measures are equal. Namely,

$$h_{\mathrm{m},\mu}(f) = h_{\mathrm{pre},\mu}(f), \text{ for every } \mu \in \mathcal{M}_f(X). \tag{4}$$

Unlike classical entropy, the variational principle for pre-image entropy is generally not valid ([19, 25]). Thus, we need to consider the variational principle under some additional conditions. Wu and Zhu [28, Theorem B], established a variational principle relating $h_{\mathrm{m}}(f)$ and $h_{\mathrm{m},\mu}(f)$ for f with *uniform separation of preimages*, i.e., there exists $\varepsilon_0 > 0$ so that $d(x, y) \leq \varepsilon_0$ and $f(x) = f(y)$ implies $x = y$. More precisely, if $f : X \to X$ is a continuous map acting on the compact metric space (X, d) and has the uniform separation of preimages property then

$$h_{\mathrm{m}}(f) = \sup_{\mu\in\mathcal{M}_f(X)} h_{\mathrm{m},\mu}(f). \tag{5}$$

However, we cannot consider the variational principle of preimage metric mean dimension under the uniform separation of preimages property, as we will prove that each continuous map with the uniform separation of preimages property has zero preimage metric mean dimension (see Theorem A).

2.4.2. *Measure-theoretic preimage metric mean dimension.* We define the following quantities via preimage structure:

$$\overline{\mathrm{Hmdim}}_{\mathrm{M,m}}(f, d) = \limsup_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|} \sup_{\mu\in\mathcal{M}_f(X)} \inf_{\xi\in\mathcal{P}_\varepsilon(X)} h_{\mathrm{m},\mu}(f, \xi) \tag{6}$$

and

$$\overline{\mathrm{Hmdim}}_{\mathrm{M,pre}}(f, d) = \limsup_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|} \sup_{\mu\in\mathcal{M}_f(X)} \inf_{\xi\in\mathcal{P}_\varepsilon(X)} h_{\mathrm{pre},\mu}(f, \xi).$$

2.5. **Main result.** The first main result may be seen as an extension of [27, Theorem B and Theorem C] to the infinite entropy setting.

Theorem A. *Suppose $f : X \to X$ is a continuous transformation on the compact metric space (X, d). Then*

$$\overline{\mathrm{Hmdim}}_{\mathrm{M,pre}}(f, d) = \overline{\mathrm{Hmdim}}_{\mathrm{M,m}}(f, d) \leq \overline{\mathrm{mdim}}_{\mathrm{M,m}}(f, d) \leq \overline{\mathrm{mdim}}_{\mathrm{M,pre}}(f, d).$$

Moreover, if in addition, $f : X \to X$ satisfying the uniform separation of preimages, then

$$\overline{\mathrm{Hmdim}}_{\mathrm{M,pre}}(f, d) = \overline{\mathrm{Hmdim}}_{\mathrm{M,m}}(f, d) = \overline{\mathrm{mdim}}_{\mathrm{M,m}}(f, d) = 0.$$

Let $X^{\mathbb{Z}}$ be the infinite product space of the compact metric space (X, d) endowed with the metric

$$\tilde{d}(\tilde{x}, \tilde{y}) = \sum_{n=0}^{\infty} \frac{d(x_{-n}, y_{-n})}{2^n} \tag{7}$$

for $\tilde{x} = (x_n)_{n\in\mathbb{Z}}$ and $\tilde{y} = (y_n)_{n\in\mathbb{Z}}$. If f is surjective, then we let (X^f, τ) be the natural extension of (X, f), where

$$X^f = \{(x_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}} : x_{n+1} = f(x_n) \text{ for all } n \in \mathbb{Z}\},$$

and τ is the left shift on X^f, which is a homeomorphism. We have a natural projection $\pi : X^f \to X, (x_n)_{n\in\mathbb{Z}} \mapsto x_0$, which is continuous and satisfies $f\circ\pi = \pi\circ\tau$.

Theorem B. *Let $f : X \to X$ be a continuous map on the compact metric space (X, d). If f is a surjective map then*

$$\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X, f, d) = \overline{\mathrm{mdim}}_{\mathrm{M}}(X^f|X, \tau^{-1}, \tilde{d}),$$

where $\overline{\mathrm{mdim}}_{\mathrm{M}}(X^f|X, \tau^{-1}, \tilde{d})$ is the conditional metric mean dimension of X^f with respect to X (see Section 4.1 for the definition).

Combining (2) and Theorem B, we have the following.

Corollary 2.3. *Under the above conditions*

$$\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X, f, d) = \mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M}}(X^f|X, \tau^{-1}, \tilde{d}).$$

The definition $\mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M}}(X^f|X, \tau^{-1}, \tilde{d})$ can be found in Section 4.

Note that the right-hand side of the equation in Corollary 2.3 is in the measure-theoretical sense, but the left-hand side of that is in the topological sense. Thus, this equation gives a different way to obtain a variational principle for the preimage metric mean dimension.

Furthermore, we introduce the *Branch preimage metric mean dimension* to provide an upper bound of mean metric dimension by preimage mean metric dimension.

Theorem C. *For any continuous surjective map $f : X \to X$ on a compact metric space,*

$$\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X, f, d) \le \overline{\mathrm{mdim}}_{\mathrm{M}}(X, f, d) \le \overline{\mathrm{mdim}}_{\mathrm{M,m}}(X, f, d) + \overline{\mathrm{mdim}}_{\mathrm{M,b}}(X, f, d),$$

and

$$\underline{\mathrm{mdim}}_{\mathrm{M,m}}(X, f, d) \le \underline{\mathrm{mdim}}_{\mathrm{M}}(X, f, d) \le \underline{\mathrm{mdim}}_{\mathrm{M,m}}(X, f, d) + \underline{\mathrm{mdim}}_{\mathrm{M,b}}(X, f, d).$$

The definition of Branch preimage entropy can be found in Section 7.

In our next result we obtain a C^0-dense set of maps on $[0, 1]$ for which we have positive preimage metric mean dimension and the variational principle holds.

Proposition 2.4. *Let $C^0([0, 1])$ be the space of continuous endomorphisms of the interval $([0, 1], d)$, where d stands for the Euclidean distance. For every $\beta \in [0, 1]$ there exists a dense subset $\mathcal{D}_\beta \subset C^0([0, 1])$ under the uniform metric $\|\cdot\|$ such that*

$$\overline{\mathrm{mdim}}_{\mathrm{M,m}}([0, 1], f, d) = \mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M,m}}([0, 1], f, d) = \beta, \quad \forall f \in \mathcal{D}_\beta.$$

Next, we consider an extension of *Bedford-McMullen carpets* [2, 17]. Let $a \ge b \ge 2$ be two natural numbers and set $A = \{0, 1, \dots, a-1\}$ and $B = \{0, 1, \dots, b-1\}$. Let $(A \times B)^{\mathbb{N}}$ be the compact metric space with a metric

$$\rho((x_n, y_n)_{n\in\mathbb{N}}, (x'_n, y'_n)_{n\in\mathbb{N}}) = 2^{-\min\{n\in\mathbb{N}:(x_n,y_n)\neq(x'_n,y'_n)\}}. \tag{8}$$

Then $((A \times B)^{\mathbb{N}}, \sigma)$ is the one-sided full-shift on the alphabet $A \times B$. Let $\pi : (A \times B)^{\mathbb{N}} \to B^{\mathbb{N}}$ be the natural projection. We also denote by $\sigma : B^{\mathbb{N}} \to B^{\mathbb{N}}$ the shift map on $B^{\mathbb{N}}$. Denote a compatible metric on $B^{\mathbb{N}}$ by

$$\rho'((y_n)_{n\in\mathbb{N}}, (y'_n)_{n\in\mathbb{N}}) = 2^{-\min\{n\in\mathbb{N}:y_n\neq y'_n\}}. \tag{9}$$

Consider $[0, 1]^{\mathbb{N}} \times [0, 1]^{\mathbb{N}}$ with the metric

$$d((x, y), (x', y')) = \sum_{n=1}^{\infty} \frac{1}{2^n} \max\{|x_n - x'_n|, |y_n - y'_n|\}, \tag{10}$$

where $x = (x_n)_{n\in\mathbb{N}}$, $x' = (x'_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}}$ and $y' = (y'_n)_{n\in\mathbb{N}} \in [0, 1]^{\mathbb{N}}$. Again, as in the previous cases, we denote by σ the shift map on $[0, 1]^{\mathbb{N}} \times [0, 1]^{\mathbb{N}}$,

$$\sigma((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = ((x_{n+1})_{n\in\mathbb{N}}, (y_{n+1})_{n\in\mathbb{N}}).$$

Let $\Omega \subset (A \times B)^{\mathbb{N}}$ be a closed non-empty set such that $\sigma(\Omega) \subset \Omega$. We define a *carpet system* $X_\Omega \subset [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$ by

$$X_\Omega = \left\{ \left(\sum_{m=1}^{\infty} \frac{x_m}{a^m}, \sum_{m=1}^{\infty} \frac{y_m}{b^m} \right) \in [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}} : (x_m, y_m) \in \Omega \text{ for all } m \in \mathbb{N} \right\}.$$

Let $\ell^\infty := \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sup_{n\geq 1} |x_n| < \infty \right\}$ be the space with the norm $\|x\|_\infty := \sup_{n\geq 1} |x_n|$ for $x = (x_n)_{n\in\mathbb{N}}$. Since $x_m \in A^{\mathbb{N}} \subset \ell^\infty$, we consider the summation $\sum_{m=1}^{\infty} \frac{x_m}{a^m}$ in ℓ^∞. Then $\sum_{m=1}^{\infty} \frac{x_m}{a^m} \in [0,1]^{\mathbb{N}}$. The same idea is applied on $\sum_{m=1}^{\infty} \frac{y_m}{b^m}$. In this way we have obtained a forward σ-invariant subset X_Ω. Then (X_Ω, σ) is a subsystem of $([0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}}, \sigma)$. Let $\Omega' = \pi(\Omega)$. Then (Ω', σ) is a subshift of $B^{\mathbb{N}}$.

In [21], Tsukamoto proved that the metric mean dimension, the topological entropy of (Ω, σ) and the topological entropy of (Ω', σ) obey the following relation

$$\mathrm{mdim}_{\mathrm{M}}(X_\Omega, \sigma, d) = \frac{h_{\mathrm{top}}(\Omega, \sigma)}{\log a} + \left(\frac{1}{\log b} - \frac{1}{\log a} \right) h_{\mathrm{top}}(\Omega', \sigma).$$

The main goal here is to extend this formula for the preimage setting.

Theorem D. *In the above setting, the preimage metric mean dimension of* (X_Ω, σ) *with respect to the metric* d*, is given by*

$$\mathrm{mdim}_{\mathrm{M,m}}(X_\Omega, \sigma, d) = \frac{h_{\mathrm{m}}(\Omega, \sigma)}{\log a} + \left(\frac{1}{\log b} - \frac{1}{\log a} \right) h_{\mathrm{m}}(\Omega', \sigma).$$

It follows from [7] that the preimage entropy h_{m} of a map $f : X \to X$ on compact metric space, is equal to its topological entropy h_{top} if f is forward expansive. It is well known that subshifts on finitely many symbols are forward expansive. Thus,

$$h_{\mathrm{m}}(\Omega, \sigma) = h_{\mathrm{top}}(\Omega, \sigma) \text{ and } h_{\mathrm{m}}(\Omega', \sigma) = h_{\mathrm{top}}(\Omega', \sigma).$$

As a corollary of Theorem D, we have that the preimage metric mean dimension of a carpet system is equal its metric mean dimension.

Corollary 2.5. *In the above setting,*

$$\mathrm{mdim}_{\mathrm{M,m}}(X_\Omega, \sigma, d) = \mathrm{mdim}_{\mathrm{M}}(X_\Omega, \sigma, d).$$

3. PROOF OF THEOREM A

We start by proving that $\mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M,m}}(f, d) = \mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M,pre}}(f, d)$. In fact it is an immediate consequence of the [27, Proposition 3.1] which asserts that $h_{\mathrm{pre},\mu}(f, \xi) = h_{\mathrm{m},\mu}(f, \xi)$ for any $\xi \in \mathcal{P}(X)$ and $\mu \in \mathcal{M}_f(X)$. This guarantees the desired equality.

Let us prove that $\mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M,m}}(f, d) \leq \overline{\mathrm{mdim}}_{\mathrm{M,m}}(f, d)$. First of all, by [27, Theorem 2.13], given $\mu \in \mathcal{M}_f(X)$, if θ is its unique ergodic decomposition, then for any $\xi \in \mathcal{P}(X)$,

$$h_{\mathrm{m},\mu}(f, \xi) = \int_{\mathcal{M}_f^e(X)} h_{\mathrm{m},\nu}(f, \xi)\, d\theta(\nu).$$

It implies that we only need to consider the f-invariant ergodic probability measures to get the inequality.

Given $\varepsilon > 0$, let $\mu \in \mathcal{M}_f^e(X)$ and $\xi = \{A_1, \ldots, A_k\} \in \mathcal{P}_\varepsilon(X)$ with $\mu(\partial\xi) = 0$. By the proof of [27, Proposition 3.4], given $\delta > 0$ there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ of X so that

$$1 - \delta < (1 + k)^{n\delta} \cdot 2^n \cdot s(n, \varepsilon, f^{-n}(z_n)) \cdot \exp\{-n(h_{\mathrm{m},\mu}(f, \xi) - 2\delta)\},$$

which gives

$$h_{\mathrm{m},\mu}(f, \xi) - 2\delta \leq \delta \log(1 + k) + \log 2 + \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} s(n, \varepsilon, f^{-n}(x)).$$

Letting $\delta \to 0$, we have

$$h_{\mathrm{m},\mu}(f,\xi) \leq \log 2 + \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} s(n,\varepsilon,f^{-n}(x)).$$

As $\mu \in \mathcal{M}_f^e(T)$ is arbitrary, it follows that

$$\overline{\mathrm{Hmdim}}_{\mathrm{M,m}}(f,d) \leq \overline{\mathrm{mdim}}_{\mathrm{M,m}}(f,d).$$

Now we suppose that f satisfies the uniform separation of preimages property with exponent $\varepsilon_0 > 0$. By [28, Corollary A.1], for any $\mu \in \mathcal{M}_f(X)$ one has that for any $\xi = \{A_1,\dots,A_k\} \in \mathcal{P}_{\varepsilon_0}(X)$,

$$h_{\mathrm{m},\mu}(f) = h_{\mathrm{m},\mu}(f,\xi) \leq \log k < \infty.$$

This implies that $h_m(f) \leq \log k < \infty$, by (5). Thus,

$$\overline{\mathrm{mdim}}_{\mathrm{M,m}}(f,d) \leq \limsup_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} \log k = 0.$$

Thus,

$$\overline{\mathrm{Hmdim}}_{\mathrm{M,pre}}(f,d) = \overline{\mathrm{Hmdim}}_{\mathrm{M,m}}(f,d) = \overline{\mathrm{mdim}}_{\mathrm{M,m}}(f,d) = 0.$$

4. PROOF OF THEOREM B

4.1. **Conditional metric mean dimension.** Let $f : X \to X$ be a continuous map on compact metric space X. Given a finite open cover of X and $m < n \in \mathbb{N}$, denote $\mathcal{U}_m^n := \bigvee_{i=m}^n f^{-i}\mathcal{U}$.

Let $f : X \to X$ and $g : Y \to Y$ be continuous maps acting on the compact metric spaces (X, d_X) and (Y, d_Y), respectively. A continuous map $\pi : X \to Y$ is called a factor map between (X, f) and (Y, g) if it is onto and $\pi \circ f = g \circ \pi$. Given $y \in Y$ and a finite open cover $\mathcal{U}$ of X, let

$$N(\mathcal{U}|y) := \min\left\{\#(\mathcal{V}) : \mathcal{V} \subset \mathcal{U}, \bigcup_{V\in\mathcal{V}} V \supset \pi^{-1}(y)\right\},$$
$$N(\mathcal{U}|Y) := \sup_{y\in Y} N(\mathcal{U}|y),$$
$$H(\mathcal{U}|Y) := \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}_0^{n-1}|Y),$$

where $\#(\mathcal{V})$ is the cardinality of elements in $\mathcal{V}$. The conditional metric mean dimension of $f : X \to X$ with respect to the factor $g : Y \to Y$ is defined by

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(X|Y, f, d_X) = \limsup_{\varepsilon\to 0} \frac{\inf_{\mathrm{diam}(\mathcal{U})<\varepsilon} H(\mathcal{U}|Y)}{|\log\varepsilon|},$$

which was presented in [26]. Furthermore, define

$$\overline{\mathrm{Hmdim}}_{\mathrm{M}}(X \mid Y, f, d_X) = \limsup_{\varepsilon\to 0} \frac{\sup_{\mu\in\mathcal{M}_f(X)} \inf_{\xi\in\mathcal{P}_\varepsilon(X)} h_\mu(f,\xi \mid Y)}{|\log\varepsilon|},$$

where $h_\mu(f, \xi \mid Y)$ is the conditional entropy of (f, ξ) with respect to (Y, g).

In what follows, for $y \in Y$, we denote by $s(d_X, f, n, \varepsilon, \pi^{-1}(y))$ the maximum cardinality of a (n, ε)-separated subset of $\pi^{-1}(y)$, and by $r(d_X, f, n, \varepsilon, \pi^{-1}(y))$ the minimum cardinality of a (n, ε)-spanning subset of $\pi^{-1}(y)$. By [26], we have the following.

Theorem 4.1. *Let $f : X \to X$ and $g : Y \to Y$ be continuous maps acting on the compact metric spaces (X, d_X) and (Y, d_Y) and $\pi : X \to Y$ a factor map between (X, f) and (Y, g). Then*

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(X|Y, f, d_X) = \limsup_{\varepsilon\to 0} \frac{\limsup_{n\to\infty} \frac{1}{n} \log \sup_{y\in Y} s(d_X, f, n, \varepsilon, \pi^{-1}(y))}{|\log\varepsilon|}.$$

Following ideas in [7, Lemma 5.1], we prove the following lemma.

Lemma 4.2. *For any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that for any $n \geq m$, if $d_{n+1}(\pi(\tilde{x}), \pi(\tilde{y})) < \varepsilon/4$ then $\tilde{d}(\tau^n(\tilde{x}), \tau^n(\tilde{y})) < \varepsilon$.*

Proof. Since (X, d_X) is a compact metric space, $M = \operatorname{diam}(X, d_X) := \max\{d_X(x, y) : x, y \in X\} < \infty$. For $\tilde{x}, \tilde{y} \in X^f$ and $m \geq 1$, we have

$$\begin{aligned}\tilde{d}(\tau^m(\tilde{x}), \tau^m(\tilde{y})) &= \sum_{i=0}^{m} \frac{1}{2^i} d(x_{m-i}, y_{m-i}) + \sum_{i=m+1}^{\infty} \frac{1}{2^i} d(x_{m-i}, y_{m-i}) \\ &\leq 2\ d_{m+1}(\pi(\tilde{x}), \pi(\tilde{y})) + \frac{M}{2^m}.\end{aligned}$$

Hence the result follows by taking $m \in \mathbb{N}$ large enough so that $\frac{M}{2^m} < \frac{\varepsilon}{2}$. □

For any $\delta > 0$, $n \in \mathbb{N}$ and $x \in X$ define

$$V_{n,\delta}(x) := \{\tilde{y} \in X^f : d_n(\pi(\tilde{y}), x) < \delta\}.$$

Lemma 4.3. *For any $\varepsilon > 0$, there exists $m = m(\varepsilon) \in \mathbb{N}$ so that for all $n \geq m$ and any $x \in X$*

$$s(\tilde{d}, \tau, n, \varepsilon, V_{n,\varepsilon/8}(x)) \leq s(\tilde{d}, \tau, m, \varepsilon, X^f),$$

where $s(\tilde{d}, \tau, n, \varepsilon, V_{n,\varepsilon/8}(x))$ denotes the maximum cardinality of an (n, ε)-separated set in $V_{n,\varepsilon/4}(x)$ with respect to $(\tilde{d}, \tau)$ and $s(\tilde{d}, \tau, m, \varepsilon, X^f)$ denotes the maximum cardinality of an (m, ε)-separated set in X^f.

Proof. If $\tilde{x}, \tilde{y} \in V_{n,\varepsilon/8}(x)$, by the triangular inequality we have $d_n(\pi(\tilde{x}), \pi(\tilde{y})) < \varepsilon/4$. Let $m \in \mathbb{N}$ given by Lemma 4.2. So, for any $n \geq m$, if $\tilde{y}, \tilde{z} \in V_{n,\varepsilon/8}(x)$, then

$$\tilde{d}(\tau^{m+j}(\tilde{x}), \tau^{m+j}(\tilde{y})) \leq \varepsilon, \ \forall j = 0, \ldots, n-m-1.$$

Thus, if in addition, $\tilde{y}, \tilde{z}$ satisfies that $\tilde{d}_m(\tilde{y}, \tilde{z}) \leq \varepsilon$, we have $\tilde{d}_m(\tilde{y}, \tilde{z}) \leq \varepsilon$ for any $n \geq m$. It gives $s(\tilde{d}, \tau, n, \varepsilon, V_{n,\varepsilon/8}(x)) \leq s(\tilde{d}, \tau, m, \varepsilon, V_{n,\varepsilon/8}(x))$. As

$$s(\tilde{d}, \tau, m, \varepsilon, V_{n,\varepsilon/8}(x)) \leq s(\tilde{d}, \tau, m, \varepsilon, X^f),$$

the result follows. □

Remark 4.4. The proof of Lemma 4.3 is based on ideas in [27, Lemma 4.3]. However, our results show $\delta > 0$ in [27, Lemma 4.3] can be taken by $\delta = \varepsilon/8$.

With the help of two lemmas above-mentioned, we have the following result.

Lemma 4.5. *For any $\varepsilon > 0$, there exists $m = m(\varepsilon) \in \mathbb{N}$ so that for all $n \geq m$ and any $x \in X$*

$$\begin{aligned}s(d, f, n, \varepsilon, f^{-n}(x)) &\leq s(\tilde{d}, \tau, n, \varepsilon, \pi^{-1}(f^{-n}(x))) \\ &\leq s(d, f, n, \varepsilon/8, f^{-n}(x)) \cdot s(\tilde{d}, \tau, m, \varepsilon, X^f).\end{aligned}$$

Proof. Note that if $\tilde{d}(\tilde{x}, \tilde{y}) < \varepsilon$, then $d(\pi(\tilde{x}), \pi(\tilde{y})) < \varepsilon$. It implies that, for $x \in X$, if E is an (n, ε)-separated set in $f^{-n}(x)$ then $\pi^{-1}(E)$ is an (n, ε)-separated set in $\pi^{-1}(f^{-n}(x))$ with cardinality bigger than that of E. Thus for any $x \in X$ and $n \in \mathbb{N}$

$$s(d, f, n, \varepsilon, f^{-n}(x)) \leq s(\tilde{d}, \tau, n, \varepsilon, \pi^{-1}(f^{-n}(x))). \tag{11}$$

Let $m = m(\varepsilon) \in \mathbb{N}$ be as in Lemma 4.3. For $x \in X$, consider $F \subset f^{-n}(x)$ a maximal $(n, \varepsilon/8)$-separated set, which is also a $(n, \varepsilon/8)$-spanning set. Thus $\pi^{-1}(f^{-n}(x)) \subset \cup_{y \in F} V_{n,\varepsilon/8}(y)$. By Lemma 4.3, we obtain

$$\begin{aligned}s(\tilde{d}, \tau, n, \varepsilon, \pi^{-1}(f^{-n}(x))) &\leq \sum_{y \in F} s(\tilde{d}, \tau, n, \varepsilon, V_{n,\varepsilon/8}(x)) \\ &\leq s(d, f, n, \varepsilon/8, f^{-n}(x)) \cdot s(\tilde{d}, \tau, m, \varepsilon, X^f).\end{aligned}$$

The proof is completed. $\square$

Now we proceed to prove Theorem B. By Lemma 4.5, we have

$$\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X}s(d,f,n,\varepsilon,f^{-n}(x))\leq\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X}s(\tilde{d},\tau,n,\varepsilon,\pi^{-1}(f^{-n}(x))).\tag{12}$$

On the other hand as $\pi^{-1}(f^{-n}(x))=\tau^{-n}(\pi^{-1}(x))$ for any $x\in X$, $A\subset(X^f,\tilde{d})$ is an (τ,n,ε)-separated subset of $\tau^{-n}(\pi^{-1}(x))$ if and only if $\tau^n(A)$ is an $(\tau^{-1},n,\varepsilon)$-separated subset of $\pi^{-1}(x)$. So

$$s(\tilde{d},\tau,n,\varepsilon,\tau^{-n}(\pi^{-1}(x)))=s(\tilde{d},\tau^{-1},n,\varepsilon,\pi^{-1}(x)),\tag{13}$$

which together with (12), implies

$$\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X}s(d,f,n,\varepsilon,f^{-n}(x))\leq\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X}s(\tilde{d},\tau^{-1},n,\varepsilon,\pi^{-1}(x)).$$

By Theorem 4.1, we obtain

$$\overline{\operatorname{mdim}}_{\mathrm{M,m}}(X,f,d)\leq\overline{\operatorname{mdim}}_{\mathrm{M}}(X^f|X,\tau^{-1},\tilde{d}).$$

For the converse inequality we notice that by Lemma 4.5 and (13), we have that

$$\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X}s(\tilde{d},\tau^{-1},n,\varepsilon,\pi^{-1}(x))\leq\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X}s(d,f,n,\varepsilon/8,f^{-n}(x)).$$

Using Theorem 4.1 again, we deduce that

$$\overline{\operatorname{mdim}}_{\mathrm{M}}(X^f|X,\tau^{-1},\tilde{d})\leq\overline{\operatorname{mdim}}_{\mathrm{M,m}}(X,f,d).$$

Now we finish the proof of Theorem B.

Proof of Corollary 2.3. By [26, Theorem 3.4], one has that

$$\mathrm{H}\overline{\operatorname{mdim}}_{\mathrm{M}}(X|Y,f,d_X)=\overline{\operatorname{mdim}}_{\mathrm{M}}(X|Y,f,d_X),$$

which together with Theorem B, implies that

$$\overline{\operatorname{mdim}}_{\mathrm{M,m}}(X,f,d)=\mathrm{H}\overline{\operatorname{mdim}}_{\mathrm{M}}(X^f|X,\tau^{-1},\tilde{d}).$$

The proof is completed. $\square$

5. PROOF OF PROPOSITION 2.4

Our construction is inspired in the one presented in [4].

Consider the sequence $(a_n)_{n\in\mathbb{N}}$ of numbers in $[0,1]$ so that $a_0=0$ and $a_n=\sum_{k=1}^{n}\frac{6}{\pi^2k^2}$. For each $n\in\mathbb{N}$ consider the interval $J_n=[a_{n-1},a_n]$ and let $\delta_n=|J_n|/3^{n+2}$. Now, define a piecewise affine map $f:[0,1]\to[0,1]$ as follows: divide the interval J_n in 3^n subintervals of same length and denote such interval as $J_n(1)=[b_n(0),b_n(1)],\dots,J_n(3^n)=[b_n(3^n-1),b_n(3^n)]$. Let $\eta_n=\frac{|J_n-\delta_n|-\delta_n}{|J_n|/3^n-\delta_n}$ and for each interval $J_n'(k):=[b_n(k)+\frac{\delta_n}{2},b_n(k+1)-\frac{\delta_n}{2}]$, $k\in\{0,1,2,\dots,3^n-1\}$, and define

$$f(x):=\begin{cases}\eta_n\left(x-(b_n(2k)-\frac{\delta_n}{2})\right)+(a_n-\frac{\delta_n}{2}), & \text{if } x\in J_n'(2k),\\ \\ -\eta_n\left(x-(b_n(2k)+\frac{\delta_n}{2})\right)+(a_n-\frac{\delta_n}{2}), & \text{if } x\in J_n'(2k+1),\end{cases}\tag{14}$$

and $f|_{[a_{n-1},a_{n-1}+\frac{\delta_n}{2}]\cup[a_n-\frac{\delta_n}{2},a_n]}$ is the identity map. In the intervals $J_n''(k):=[b_n(k)-\frac{\delta_n}{2},b_n(k)+\frac{\delta_n}{2}]$, if k is odd, define f as an increasing affine map in $[b_n(k)-\frac{\delta_n}{2},b_n(k)]$ so that $f(b_n(k)-\frac{\delta_n}{2})=a_n-\frac{\delta_n}{2}$ and $f(b_n(k))=a_n$, and a decreasing affine map in $[a_k,b_n(k)+\frac{\delta_n}{2}]$ so that $f(b_n(k))=a_n$ and $f(b_n(k)+\frac{\delta_n}{2})=a_n-\frac{\delta_n}{2}$; if k is even, define f, similarly (see Figure 1).

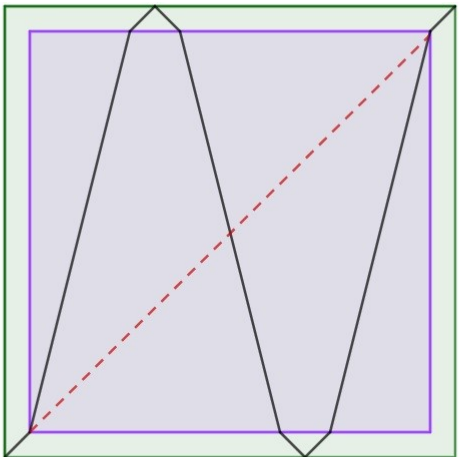

FIGURE 1. Local construction of $f|_{J_1}$.

Claim: $\overline{\text{mdim}}_{\text{M,m}}([0,1], f, d) = 1$.

Let us define $G_n = [a_{n-1} + \frac{\delta_n}{2}, a_n - \frac{\delta_n}{2}]$ and $G_n^\infty = \bigcap_{j=0}^\infty f^{-j}(G_n)$. We notice that, by the definition of f, $\overline{\text{mdim}}_{\text{M,m}}([0,1] \setminus \bigcup_{n=1}^\infty G_n^\infty, f, d) = 0$. Moreover, by [7, Corollary 2.4], as $f|_{G_n^\infty}$ has uniform separation of preimage property with separation constant δ_n, there exists $x_n \in J_n$ so that $f^{-N}(x_n)$ is (N, δ_n)-separated set with $\#(f^{-N}(x_n)) \geq (3^n)^N$. It gives

$$\sup_{x \in [0,1]} s(f, N, \delta_n, f^{-N}(x)) \geq s(f, N, \delta_n, f^{-N}(x)) \geq (3^n)^N \tag{15}$$

and then, as $|J_n| = \frac{6}{\pi^2 n^2}$, we obtain

$$\begin{aligned}
\overline{\text{mdim}}_{\text{M,m}}([0,1], f, d) &= \limsup_{\varepsilon \to 0} \frac{\limsup\limits_{N \to \infty} \frac{1}{N} \log \sup\limits_{x \in [0,1]} s(f, N, \varepsilon, f^{-N}(x))}{-\log \varepsilon} \\
&\geq \limsup_{n \to \infty} \frac{\limsup\limits_{N \to \infty} \frac{1}{N} \log \sup\limits_{x \in [0,1]} s(f, N, \delta_n, f^{-N}(x))}{-\log \delta_n} \\
&\geq \limsup_{n \to \infty} \frac{n \log 3}{-\log 6 + 2 \log n + 2 \log \pi + (n+2) \log 3} \\
&= 1.
\end{aligned}$$

Using the fact that $\overline{\text{mdim}}_{\text{M}}([0,1], f, d) = 1$ [1], we obtain that $\overline{\text{mdim}}_{\text{M,m}}([0,1], f, d) = 1$.

Based on the previous construction we have $f|_{G_n}$ has the uniform separation of preimages property with exponent δ_n. So, by [28, Corollary A.1] for $\nu \in \mathcal{M}_{f|_{G_n^\infty}}(G_n^\infty)$ we have

$$h_{m,\nu}(f|_{G_n^\infty}, \xi) = h_{m,\nu}(f|_{G_n^\infty}) \leq h_{m,\tilde{\nu}}(f, \bar{\xi}),$$

where ξ is a Borel partition of G_n^∞ with $\text{diam}(\xi) < \varepsilon < \delta_n$, $\bar{\xi}$ is a Borel partition of $[0,1]$ so that any element of ξ is contained in a element of $\bar{\xi}$ and $\tilde{\nu}$ is the push forward of ν by the inclusion map $i : G_n^\infty \to [0,1]$. Moreover, by [28, page 20], if

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[1] This can be obtained by an argument similar to that of [22, Proposition 8].

we take ξ as before and satisfying $\mu(\partial\xi)=0$ we obtain $\mu\in\mathcal{M}_{f|_{G_n^\infty}}(G_n^\infty)$ so that

$$h_m(f|_{G_n^\infty},\varepsilon)\le h_{m,\tilde{\mu}}(f,\bar{\xi}).$$

By (15) we have that $s(f|_{G_n^\infty},N,\delta_n,f^{-N}(x))\ge(3^n)^N$, for some $x\in G_n^\infty$. Hence

$$n\log 3\le h_m(f|_{G_n^\infty},\delta_n)\le h_{m,\tilde{\mu}}(f,\bar{\xi}).$$

It implies

$$1=\overline{\mathrm{mdim}}_{\mathrm{M}}([0,1],f,d)=\overline{\mathrm{mdim}}_{\mathrm{M,m}}([0,1],f,d)=\mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M,m}}(f,d).$$

Now for $\beta\in(0,1)$ we define $J_n=[a_{n-1}^\beta,a_n^\beta]$, where the the general term of the sequence $(a_n)_{n\in\mathbb{N}}$ is determined by

$$n\in\mathbb{N}\mapsto a_n^\beta=\sum_{k=0}^{n}C(\beta)3^{k(1-1/\beta)},$$

where $C(\beta)=(\sum_{k=1}^{\infty}3^{k(1-1/\beta)})^{-1}$. Divide J_n^β in 3^n subintervals, denoted by $J_n^\beta(1)=[b_n^\beta(0),b_n^\beta(1)]$,..., $J_n^\beta(3^n)=[b_n^\beta(3^{n-1}),b_n^\beta(3^n)]$, of length $C(\beta)3^{-n/\beta}$. Define

$$\delta_n=C(\beta)3^{-(n+2)/\beta}.$$

Then we define the intervals $(J_n^\beta)'(k)=[b_n^\beta(k-1)+\delta_n/2,b_n^\beta(k)-\delta_n/2]$, $k=1,\dots,3^n-1$ and $(J_n^\beta)''(k)=[b_n^\beta(k-1)-\delta_n/2,b_n^\beta(k-1)+\delta_n/2]$, $k=2,\dots,3^n$. Let $G_n^\beta=[a_{n-1}^\beta+\frac{\delta_n}{2},a_n^\beta-\frac{\delta_n}{2}]$ and $G_n^\infty=\bigcap_{j=0}^\infty f^{-j}(G_n^\beta)$. By a construction similar to that of the case $\beta=1$, we can define f_β on $[0,1]$. Then

$$\begin{aligned}\overline{\mathrm{mdim}}_{\mathrm{M,m}}([0,1],f_\beta,d)&=\limsup_{\varepsilon\to0}\frac{\limsup\limits_{N\to\infty}\dfrac{1}{N}\log\sup\limits_{x\in[0,1]}s(f_\beta,N,\varepsilon,f_\beta^{-N}(x))}{-\log\varepsilon}\\&\ge\limsup_{n\to\infty}\frac{\limsup\limits_{N\to\infty}\dfrac{1}{N}\log\sup\limits_{x\in[0,1]}s(f_\beta,N,\delta_n,f_\beta^{-N}(x))}{-\log\delta_n}\\&\ge\limsup_{n\to\infty}\frac{n\log 3}{-\log C(\beta)+(n+2)\frac{1}{\beta}\log 3}\\&=\beta.\end{aligned}$$

Finally, to prove that, for each $\beta\in[0,1]$, there exists a dense subset of $C^0([0,1])$ with preimage metric mean dimension β and satisfying the variational principle we notice that for $\beta=0$ the set of C^1 maps on $[0,1]$ is C^0-dense in $C^0([0,1])$. For $0<\beta\le1$, we take $f\in C^0([0,1])$ and $\varepsilon>0$ and show that there exists $h\in C^0([0,1])$ with $\|f-h\|<\varepsilon$ and $\overline{\mathrm{mdim}}_{\mathrm{M}}([0,1],h,d)=\beta$.

First of all, we take a C^1 map h_1 so that $\|h_1-f\|<\varepsilon/3$. For a fixed point P of h_1, take $h_2\in C^0([0,1])$ so that $\|h_1-h_2\|<\varepsilon/3$ and its set of fixed points in a small neighbourhood of P consists of an interval I centered in P. We notice that h_2 can be taken as C^1-map at points except, possibly, in the extremes of I. Let $J'\subsetneq J''\subsetneq I$, where J',J'' are subintervals of diameter small than $\varepsilon/3$, and consider a C^1-map χ on $[0,1]$ so that $\chi\equiv1$ on J' and $\chi\equiv0$ on $[0,1]\backslash J''$.

If T_λ denotes the homothety of parameter $\lambda\in(0,1)$ and $|J'|$ stand for the diameter of the interval J', since $\{\chi,1-\chi\}$ is a partition of unity, the map

$$h_3:=h_{3,\beta}=(1-\chi)\cdot h_2+\chi\cdot T_{|J'|}\circ f_\beta\circ T_{|J'|^{-1}}\tag{16}$$

is continuous, coincides with h_2 on $[0,1]\backslash J''$ and is linearly conjugate to f_β on the interval J'. Moreover, as h_2 is uniform continuous, h_3 may be chosen so that $\|h_3-h_2\|<\varepsilon/3$ provided that J',J'' are small enough. By triangular inequality we have that $\|h_3 - f\| < \varepsilon$ and, since all maps in the combination (16) but f_β are smooth (except possibly at two points), then

$$\beta = \overline{\mathrm{mdim}}_{\mathrm{M,m}}([0,1], f_\beta, d) = \overline{\mathrm{mdim}}_{\mathrm{M,m}}([0,1], h_3, d),$$

which proves Proposition 2.4.

6. Proof of Theorem D

In order to prove Theorem D we need to introduce some notations. Given $a \geq b \geq 2$, set

$$\omega = \log_a b = \frac{\log b}{\log a}.$$

Then $0 < w \leq 1$. For any $N \in \mathbb{N}$, we denote by $\Omega|_N$ and $\Omega'|_N$ the images of Ω and Ω' under the projections

$$(A \times B)^{\mathbb{N}} \to A^N \times B^N, \quad \left((u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}\right) \mapsto \left((u_1, \ldots, u_N), (v_1, \ldots, v_N)\right),$$
$$B^{\mathbb{N}} \to B^N, \quad (v_n)_{n\in\mathbb{N}} \mapsto (v_1, \ldots, v_N).$$

We also define $X_\Omega|_N$ as the image of X_Ω under the projection

$$[0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}} \to [0,1]^N \times [0,1]^N, ((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) \mapsto ((x_1, \ldots, x_N), (y_1, \ldots, y_N)).$$

Then

$$X_\Omega|_N = \left\{ \left(\sum_{m=1}^{\infty} \frac{x_m}{a^m}, \sum_{m=1}^{\infty} \frac{y_m}{b^m} \right) \in [0,1]^N \times [0,1]^N : (x_m, y_m) \in \Omega|_N \text{ for all } m \in \mathbb{N} \right\}.$$

Let $(x, y) \in (\Omega|_N)^{\mathbb{N}}$, where $x = (x_m)_{m\in\mathbb{N}}$ and $y = (y_m)_{m\in\mathbb{N}}$ with $x_m \in A^N, y_m \in B^N$ and $(x_m, y_m) \in \Omega|_N$. For $N, M \in \mathbb{N}$, we define a subset $Q_{N,M}(x, y) \subset X_\Omega|_N$ by

$$Q_{N,M}(x,y) = \left\{ \left(\sum_{m=1}^{\infty} \frac{x'_m}{a^m}, \sum_{m=1}^{\infty} \frac{y'_m}{b^m} \right) : \begin{array}{c} (x'_m, y'_m) \in \Omega|_N \text{ for all } m \in \mathbb{N} \text{ with} \\ x'_m = x_m (1 \leq m \leq [\omega M]) \text{ and} \\ y'_m = y_m (1 \leq m \leq M) \end{array} \right\},$$

where $[\omega M]$ is the largest integer not greater than ωM. The set $Q_{N,M}(x, y)$ depends only on the coordinates $x_1, \ldots, x_{[\omega M]}, y_1, \ldots, y_M$. Thus we also denote it by $Q_{N,M}\left(x_1, \ldots, x_{[\omega M]}, y_1, \ldots, y_M\right)$. From (5.6) in [21], we have

$$\operatorname{diam}\left(Q_{N,M}(x,y), \|\cdot\|_\infty\right) < ab^{-M}, \tag{17}$$

where $\|\cdot\|_\infty$ is the ℓ^∞-distance on $X_\Omega|_N \subset \mathbb{R}^{2N}$. Denote by $s(F|_N, \|\cdot\|_\infty, \varepsilon)$ the maximal cardinality of ε-separated sets in the subset $F|_N$ of $X_\Omega|_N$.

Lemma 6.1. *Let $\varepsilon > 0$. For any $\mathbf{x} \in X_\Omega$, and $N \in \mathbb{N}$, we have that*

$$s(\sigma^{-N}(\mathbf{x})|_N, \|\cdot\|_\infty, 2\varepsilon) \leq \mathrm{s}(\sigma^{-N}(\mathbf{x}), d_N, \varepsilon) \leq s(\sigma^{-N}(\mathbf{x})|_N, \|\cdot\|_\infty, \varepsilon/2),$$

where d is the metric defined by (10).

Proof. Following ideas in [21, Lemma 5.5], we have that for any $\mathbf{y}, \mathbf{z} \in X_\Omega$,

$$\|\mathbf{y}|_N - \mathbf{z}|_N\|_\infty \leq 2 d_N(\mathbf{y}, \mathbf{z}), \tag{18}$$

and

$$d_N(\mathbf{y}, \mathbf{z}) \leq \|\mathbf{y}|_{N+L} - \mathbf{z}|_{N+L}\|_\infty + \varepsilon/2, \tag{19}$$

where $L \in \mathbb{N}$ satisfies that $\sum_{n>L} 2^{-n} < \varepsilon/2$. Fix any $\mathbf{x} \in X_\Omega$. By (18), we have

$$s(\sigma^{-N}(\mathbf{x})|_N, \|\cdot\|_\infty, 2\varepsilon) \leq \mathrm{s}(\sigma^{-N}(\mathbf{x}), d_N, \varepsilon).$$

By (19), we have

$$\mathrm{s}(\sigma^{-N}(\mathbf{x}), d_N, \varepsilon) \leq s(\sigma^{-N}(\mathbf{x})|_{N+L}, \|\cdot\|_\infty, \varepsilon/2) = s(\sigma^{-N}(\mathbf{x})|_N, \|\cdot\|_\infty, \varepsilon/2).$$

The proof is completed. □

It is well known that (Ω, σ) and (Ω', σ) satisfy uniform separation of preimages. Denote by $\gamma, \gamma' > 0$ the uniform separation exponents with respect to the metrics defined by (8) and (9), respectively. Then for any $\eta \in (0, \gamma)$ and $\eta' \in (0, \gamma')$,

$$\#(\sigma^{-N}(x, y)) = s(\rho_N, \sigma^{-N}(x, y), \eta) \text{ and } \#(\sigma^{-N}(y)) = s(\rho'_N, \sigma^{-N}(y), \eta'). \tag{20}$$

Lemma 6.2. *For any $N, M \in \mathbb{N}$, we have that for any $\eta \in (0, \gamma)$ and $\eta' \in (0, \gamma')$,*

$$\begin{aligned}
&\sup_{\mathbf{x}\in X_\Omega} s(\sigma^{-N}(\mathbf{x})|_N, \|\cdot\|_\infty, ab^{-M}) \\
&\le \left(\sup_{(x,y)\in\Omega} s(\rho_N, \sigma^{-N}(x, y), \eta)\right)^{[\omega M]} \cdot \left(\sup_{y\in\Omega'} s(\rho'_N, \sigma^{-N}(y), \eta')\right)^{M-[\omega M]} \\
&\le \sup_{\mathbf{x}\in X_\Omega} s(\sigma^{-N}(\mathbf{x})|_N, \|\cdot\|_\infty, b^{-M}).
\end{aligned} \tag{21}$$

Proof. Given $\mathbf{x} = (\sum_{m=1}^{\infty} \frac{x_m^*}{2^m}, \sum_{m=1}^{\infty} \frac{y_m^*}{2^m}) \in X_\Omega$, by an argument similar to that of [21, Lemma 5.6], one has that $\sigma^{-N}(\mathbf{x})$ is the union of all

$$Q_{N,M}(x_1, \dots, x_{[\omega M]}, y_1, \dots, y_M),$$

where $(x_m, y_m) \in \sigma^{-N}(x_m^*, y_m^*)|_N$ for $1 \le m \le [\omega M]$, and $y_m \in \sigma^{-N}(y_m^*)|_N$ for $[\omega M] + 1 \le m \le M$.

By (17), $\operatorname{diam}(Q_{N,M}(x, y)) < ab^{-M}$ for any $(x, y) \in \Omega$, it follows that for any $\eta \in (0, \gamma)$ and $\eta' \in (0, \gamma')$,

$$\begin{aligned}
s(\sigma^{-N}(\mathbf{x})|_N, \|\cdot\|_\infty, ab^{-M}) &\le \prod_{m=1}^{[\omega M]} \#(\sigma^{-N}(x_m^*, y_m^*)|_N) \cdot \prod_{m=[\omega M]+1}^{M} \#(\sigma^{-N}(y_m^*)|_N) \\
&= \prod_{m=1}^{[\omega M]} \#(\sigma^{-N}(x_m^*, y_m^*)) \cdot \prod_{m=[\omega M]+1}^{M} \#(\sigma^{-N}(y_m^*)) \\
&\overset{(20)}{=} \prod_{m=1}^{[\omega M]} s(\rho_N, \sigma^{-N}(x_m^*, y_m^*), \eta) \cdot \prod_{m=[\omega M]+1}^{M} s(\rho'_N, \sigma^{-N}(y_m^*), \eta') \\
&\le \left(\sup_{(x,y)\in\Omega} s(\rho_N, \sigma^{-N}(x, y), \gamma)\right)^{[\omega M]} \cdot \left(\sup_{y\in\Omega'} s(\rho'_N, \sigma^{-N}(y), \gamma')\right)^{M-[\omega M]},
\end{aligned}$$

proving the first inequality.

Now we prove the second inequality. For any $(x, y) \in \Omega$ and $y' \in \Omega'$, fix $(s, t) \in \sigma^{-N}(x, y)|_N$. For any $v \in \pi(\sigma^{-N}(x, y'))|_N$, we choose $l(v) \in A^N$ such that $(l(v), v) \in \sigma^{-N}(x, y')|_N$. For $(x_m, y_m) \in \sigma^{-N}(x, y)|_N$, $(1 \le m \le [\omega M])$ and $y_m \in \pi(\sigma^{-N}(x, y'))|_N$, $([\omega M] + 1 \le m \le M)$, we set

$$\begin{aligned}
p(x_1, \dots, &x_{[\omega M]}, y_1, \dots, y_M) \\
&= \left(\sum_{m=1}^{[\omega M]} \frac{x_m}{a^m} + \sum_{m=[\omega M]+1}^{M} \frac{l\,(y_m)}{a^m} + \sum_{m=M+1}^{\infty} \frac{s}{a^m}, \sum_{m=1}^{M} \frac{y_m}{b^m} + \sum_{m=M+1}^{\infty} \frac{t}{b^m}\right),
\end{aligned}$$

which is a point in $X_\Omega|_N$. For

$$(x_1, \dots, x_{[\omega M]}, y_1, \dots, y_M) \ne (x'_1, \dots, x'_{[\omega M]}, y'_1, \dots, y'_M),$$

we have

$$\begin{aligned}
\|p(x_1, \dots, x_{[\omega M]}, y_1, \dots, y_M) - p(x'_1, \dots, x'_{[\omega M]}, y'_1, \dots, y'_M)\|_\infty &\ge \min\{a^{-[\omega M]}, b^{-M}\} \\
&= b^{-M}.
\end{aligned}$$

Let

$$\mathbf{x} = \left(\sum_{m=1}^{\infty} \frac{x}{2^m}, \sum_{m=1}^{[\omega M]} \frac{y}{2^m} + \sum_{m=[\omega M]+1}^{M} \frac{y'}{2^M} + \sum_{m=M+1}^{\infty} \frac{y}{2^m} \right) \in X_\Omega.$$

Then each $p(x_1, \dots, x_{[\omega M]}, y_1, \dots, y_M) \in \sigma^{-N}(\mathbf{x})$. Thus,

$$s(\sigma^{-N}\mathbf{x}|_N, \|\cdot\|_\infty, b^{-M}) \geq \#(\sigma^{-N}(x,y)|_N)^{[\omega M]} \cdot \#(\sigma^{-N}(y'_m)|_N)^{M-[\omega M]}.$$

For any $\eta \in (0, \gamma)$ and $\eta' \in (0, \gamma')$, by (20), we have that

$$s(\sigma^{-N}\mathbf{x}|_N, \|\cdot\|_\infty, b^{-M}) \geq s(\rho_N, \sigma^{-N}(x,y), \eta)^{[\omega M]} \cdot s(\rho'_N, \sigma^{-N}(y'), \eta')^{M-[\omega M]}.$$

As $(x,y) \in \Omega$ and $y' \in \Omega'$ are arbitrary, the second inequality is proved. □

Now we proceed to prove Theorem D. Applying Lemmas 6.1 and 6.2, we have that

$$\begin{aligned}
&\mathrm{mdim}_{\mathrm{m,M}}(X_\Omega, \sigma, d) \\
&= \lim_{M\to\infty} \lim_{N\to\infty} \frac{1}{NM\log b} \log \Bigg(\sup_{(x,y)\in\Omega} s(d_N, \sigma^{-N}(x,y), \gamma/2))^{[\omega M]} \\
&\qquad\qquad \cdot (\sup_{y\in\Omega'} s(d_N, \sigma^{-N}(y), \gamma'/2))^{M-[\omega M]} \Bigg) \\
&= \lim_{M\to\infty} \lim_{N\to\infty} \frac{1}{NM\log b} \Bigg([\omega M] \log \sup_{(x,y)\in\Omega} s(d_N, \sigma^{-N}(x,y), \gamma/2) \\
&\qquad\qquad + (M - [\omega M]) \log \sup_{y\in\Omega'} s(d_N, \sigma^{-N}(y), \gamma'/2) \Bigg) \\
&= \lim_{M\to\infty} \frac{1}{M\log b} ([\omega M] h_m(\Omega, \sigma) + (M - [\omega M]) h_{\mathrm{m}}(\Omega', \sigma)) \\
&= \frac{\omega}{\log b} h_{\mathrm{m}}(\Omega, \sigma) + \frac{1-\omega}{\log b} h_{\mathrm{m}}(\Omega', \sigma) \\
&= \frac{h_{\mathrm{m}}(\Omega, \sigma)}{\log a} + \left(\frac{1}{\log b} - \frac{1}{\log a} \right) h_{\mathrm{m}}(\Omega', \sigma).
\end{aligned}$$

The proof of Theorem D is completed.

7. Branch preimage metric mean dimension

In this section, we introduce the Branch preimage metric mean dimension, and prove Theorem C. Recall that for any compact metric space (X, d), there exists an associated Hausdorff metric $\mathfrak{H}d$, which makes the space $K(X)$ of nonempty closed subset of X into a compact metric space: for any $F_1, F_2 \in K(X)$,

$$\mathfrak{H}d(F_1, F_2) = \max\{ \sup_{x\in F_1} \inf_{z\in F_2} d(x,z), \sup_{x\in F_2} \inf_{z\in F_1} d(x,z) \}.$$

Given $f : X \to X$ a continuous surjection, define a sequence of branch metrics on X via

$$d_n^{\mathrm{b}}(x,z) = \mathfrak{H}d_n(f^{-n}(x), f^{-n}(z)) \text{ for any } n \in \mathbb{N}.$$

That is, $x \in X$ is "branch close" to $x' \in X$ if every branch at x is shadowed by some branch at x', and vice-versa.

Given $n \in \mathbb{N}$ and $\varepsilon > 0$, denote by $s_{\mathrm{b}}(f, n, \varepsilon)$ the maximal cardinality of all (n, ε)-separated subsets of X with respect to the metric d_n^{b}. The *branch preimage entropy* [13] is defined by

$$h_{\mathrm{b}}(f, X) = \lim_{\varepsilon\to 0} h_{\mathrm{b}}(f, X, \varepsilon),$$

where $h_{\mathrm{b}}(f,X,\varepsilon)=\lim_{n\to\infty}\frac{1}{n}\log s_{\mathrm{b}}(f,n,\varepsilon)$. Following this idea, the *upper branch preimage metric mean dimension* of f with respect to d is given by

$$\overline{\mathrm{mdim}}_{\mathrm{M,b}}(X,f,d)=\limsup_{\varepsilon\to 0}\frac{h_{\mathrm{b}}(f,X,\varepsilon)}{|\log\varepsilon|}.$$

Similarly, the *lower branch preimage metric mean dimension* of f with respect to d is given by

$$\underline{\mathrm{mdim}}_{\mathrm{M,b}}(X,f,d)=\liminf_{\varepsilon\to 0}\frac{h_{\mathrm{b}}(f,X,\varepsilon)}{|\log\varepsilon|}.$$

Then we have the following inequality relating preimage metric mean dimension, branch preimage metric mean dimension, and metric mean dimension.

Theorem 7.1. *For any continuous surjective map $f:X\to X$ on a compact metric space,*

$$\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X,f,d)\le\overline{\mathrm{mdim}}_{\mathrm{M}}(X,f,d)\le\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X,f,d)+\overline{\mathrm{mdim}}_{\mathrm{M,b}}(X,f,d),$$

and

$$\underline{\mathrm{mdim}}_{\mathrm{M,m}}(X,f,d)\le\underline{\mathrm{mdim}}_{\mathrm{M}}(X,f,d)\le\underline{\mathrm{mdim}}_{\mathrm{M,m}}(X,f,d)+\underline{\mathrm{mdim}}_{\mathrm{M,b}}(X,f,d).$$

Proof. The inequalities

$$\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X,f,d)\le\overline{\mathrm{mdim}}_{\mathrm{M}}(X,f,d)\text{ and }\underline{\mathrm{mdim}}_{\mathrm{M,m}}(X,f,d)\le\underline{\mathrm{mdim}}_{\mathrm{M}}(X,f,d)$$

are clear from definitions. Now we prove other inequalities. Given $\varepsilon>0$ and $n\in\mathbb{N}$, by the proof of [13, Theorem 3.1] we have that

$$r(f,n,\varepsilon)\le\sup_{x\in X}s(n,\varepsilon/3,f^{-n}(x))\cdot s_{\mathrm{b}}(f,n,\varepsilon/3),$$

which implies that

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(X,f,d)\le\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X,f,d)+\overline{\mathrm{mdim}}_{\mathrm{M,b}}(X,f,d)$$

and

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(X,f,d)\le\underline{\mathrm{mdim}}_{\mathrm{M,m}}(X,f,d)+\underline{\mathrm{mdim}}_{\mathrm{M,b}}(X,f,d).$$

The proof is completed. □

Corollary 7.2. *For any continuous surjective map $f:X\to X$ on a compact metric space, if $\overline{\mathrm{mdim}}_{\mathrm{M,b}}(X,f,d)=0$, then*

$$\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X,f,d)=\overline{\mathrm{mdim}}_{\mathrm{M}}(X,f,d).$$

The following result of a shift map can be proved by the same argument in [18, Section 5.1].

Example 7.3. Let (X,d) be a compact metric space. Then

$$h_{\mathrm{b}}(\sigma,X^{\mathbb{N}})=\overline{\mathrm{mdim}}_{\mathrm{M,b}}(X^{\mathbb{N}},\sigma,\tilde{d})=0,$$

where $\tilde{d}$ is defined by (7). Thus,

$$\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X^{\mathbb{N}},\sigma,\tilde{d})=\overline{\mathrm{mdim}}_{\mathrm{M}}(X^{\mathbb{N}},\sigma,\tilde{d}).$$

8. Examples

8.1. **Shift maps on compact alphabets.** We further study Example 7.3, which holds the variational principle for the preimage metric mean dimension. Given $\varepsilon > 0$, define $N(\varepsilon)$ as the maximal cardinality of an ε-separated set in (X, d). The upper Minkowski dimension or upper box dimension of X is defined as

$$\overline{\dim}_{\mathrm{B}}(X, d) = \limsup_{\epsilon \to 0} \frac{N(\epsilon)}{|\log \epsilon|}.$$

Proposition 8.1. *Let (X, d) be a compact metric space. Then*

$$\begin{aligned} \mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X^{\mathbb{N}}, \sigma, \tilde{d}) = \overline{\mathrm{mdim}}_{\mathrm{M,m}}(X^{\mathbb{N}}, \sigma, \tilde{d}) &= \overline{\dim}_{\mathrm{B}}(X, d) \\ &= \overline{\mathrm{mdim}}_{\mathrm{M}}(X^{\mathbb{N}}, \sigma, \tilde{d}) \\ &= \mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M}}(X^{\mathbb{N}}, \sigma, \tilde{d}). \end{aligned}$$

Proof. It follows from [22, Theorem 5] that

$$\overline{\dim}_{\mathrm{B}}(X, d) = \overline{\mathrm{mdim}}_{\mathrm{M}}(X^{\mathbb{N}}, \sigma, \tilde{d}) = \mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M}}(X^{\mathbb{N}}, \sigma, \tilde{d}).$$

Since

$$\begin{aligned} \mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M}}(X^{\mathbb{N}}, \sigma, \tilde{d}) = \overline{\mathrm{mdim}}_{\mathrm{M}}(X^{\mathbb{N}}, \sigma, \tilde{d}) &\geq \overline{\mathrm{mdim}}_{\mathrm{M,m}}(X^{\mathbb{N}}, \sigma, \tilde{d}) \\ &\geq \mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X^{\mathbb{N}}, \sigma, \tilde{d}), \end{aligned}$$

it suffices to prove that

$$\mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M}}(X^{\mathbb{N}}, \sigma, \tilde{d}) = \mathrm{H}\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X^{\mathbb{N}}, \sigma, \tilde{d}).$$

Indeed, this is obtained by $\mathcal{B}^- = \{\emptyset, X\}$. To see this, we consider any $\emptyset \neq A \in \mathcal{B}^-$. For any $n \in \mathbb{N}$, since $A \in \sigma^{-n}\mathcal{B}$, there exists a subset $C \in \mathcal{B}$ such that $A = \sigma^{-n}C = X^n \times C$. As $n \in \mathbb{N}$ is arbitrary, it follows that $A = X^{\mathbb{N}}$. Thus, $\mathcal{B}^- = \{\emptyset, X\}$. □

Remark 8.2. Since for any $w \in X^{\mathbb{N}}$ we have that $s(n, \varepsilon, f^{-n}(w)) \geq \lfloor \frac{1}{\varepsilon} \rfloor^{n+k}$, where $k = k(\varepsilon)$, we have the following

$$\begin{aligned} \overline{\mathrm{mdim}}_{\mathrm{M,m}}(X^{\mathbb{N}}, \sigma, \tilde{d}) &= \limsup_{\varepsilon \to 0} \frac{\sup_{z \in X^{\mathbb{N}}} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, f^{-n}(z))}{-\log \varepsilon} \\ &= \limsup_{\varepsilon \to 0} \frac{\limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, f^{-n}(w))}{-\log \varepsilon}, \end{aligned}$$

for any $w \in X^{\mathbb{N}}$.

8.2. **Self-similar system.** Let $\ell^\infty = \left\{ (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sup_{n \geq 1} |x_n| < \infty \right\}$ be the space of bounded sequences with the norm $\|x\|_\infty = \sup_{n \geq 1} |x_n|$ for $x = (x_n)_{n \in \mathbb{N}}$. We always assume that ℓ^∞ is endowed with the weak* topology as the dual space of ℓ^1. We define the shift map $\sigma : \ell^\infty \to \ell^\infty$ by

$$\sigma\left((x_n)_{n \in \mathbb{N}} \right) = (x_{n+1})_{n \in \mathbb{N}},$$

which is continuous with respect to the weak* topology. We will consider a certain self-similar set of ℓ^∞ which is σ-invariant.

Let Ω be a compact metric space with a metric ρ, and $T : \Omega \to \Omega$ be a continuous map. Suppose that for each $\omega \in \Omega$, there exists a point $a(\omega) = (a(\omega)_n)_{n \in \mathbb{N}} \in \ell^\infty$ so that the map $a : \Omega \to \ell^\infty$, $\omega \mapsto a(\omega)$ is continuous with respect to the weak* topology of ℓ^∞, and $\sigma(a(\omega)) = a(T(\omega))$ for any $\omega \in \Omega$. Since Ω is compact, we have

$$\sup_{\omega \in \Omega} \|a(\omega)\|_\infty < \infty.$$

Fix $c \in \mathbb{R}$ with $0 < c < 1$. Given $\omega \in \Omega$, we define a contracting similarity transformation $S_\omega : \ell^\infty \to \ell^\infty$ by

$$S_\omega(x) = cx + a(\omega).$$

Then $\sigma\left(S_\omega(x)\right) = S_{T\omega}(\sigma(x))$. Tsukamoto [21, Proposition 4.1] proved that

$$X = \left\{ \sum_{k=0}^{\infty} c^k a\left(\omega_k\right) \mid \omega_k \in \Omega,\ k \in \mathbb{Z}_+ \right\} \tag{22}$$

is the unique non-empty compact σ-invariant, i.e. $\sigma(X) \subset X$, subset of ℓ^∞ satisfying

$$X = \bigcup_{\omega\in\Omega} S_\omega(X).$$

Define a metric d on X by

$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - y_n|.$$

The dynamical system (X, σ) is called a self-similar system defined by the family of contracting similarity transformations $\{S_\omega\}_{\omega\in\Omega}$. Moreover, Tsukamoto [21, Theorem 4.3] provided an upper bound of mean metric dimension of this system by topological entropy of (Ω, T). Now we prove the corresponding result also holds for preimage mean metric dimension.

Theorem 8.3. *Under the above setting, the self-similar system X satisfies that*

$$\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X, \sigma, d) \le \frac{h_{\mathrm{m}}(T, \Omega)}{\log(1/c)}.$$

Proof. Given $x \in X$, by (22), it can be denoted as $x = \sum_{k=0}^{\infty} c^k a(\omega_k)$. Note that for any $k \in \mathbb{Z}_+$,

$$\sigma^{-1}(a(\omega_k)) = \{a(\omega_k') : \omega_k' \in T^{-1}(\omega_k)\},$$

and hence

$$\sigma^{-1}(x) = \left\{ \sum_{k=0}^{\infty} c^k a(\omega_k') : \omega_k' \in T^{-1}(\omega_k),\ k \in \mathbb{Z}_+ \right\}.$$

Similarly, for any $N \in \mathbb{Z}_+$,

$$\sigma^{-N}(x) = \left\{ \sum_{k=0}^{\infty} c^k a(\omega_k') : \omega_k' \in T^{-N}(\omega_k),\ k \in \mathbb{Z}_+ \right\}.$$

By the continuity of $a(\cdot)$ and the compactness of X, for any $\varepsilon \in (0, 1)$, there exists $\delta > 0$ such that $\rho(\omega, \omega') < \delta$ implies that $d(a(\omega), a(\omega')) < \varepsilon/6$, and hence for any $N \in \mathbb{Z}_+$, $\rho_N(\omega, \omega') < \delta$ implies that

$$d_N(a(\omega), a(\omega')) < \varepsilon/6. \tag{23}$$

Let $M = \mathrm{diam}(X, d)$. Then $M = \mathrm{diam}(X, d_N)$ for each $N \in \mathbb{N}$. Given any $k, N \in \mathbb{Z}_+$, let E_k^N be a (N, δ)-spanning set of $T^{-N}(\omega_k)$ with respect to ρ with the minimal cardinality $r(N, \delta, T^{-N}(\omega_k))$. Choose $K \in \mathbb{Z}_+$ such that $Mc^K < \frac{\varepsilon}{6} \le Mc^{K-1}$. Then $\sum_{k=K}^{\infty} c^k M \le \frac{\varepsilon}{6(1-c)}$. Fix $\omega_K^*, \omega_{K+1}^*, \ldots \in \Omega$, and let

$$F_k^N = \left\{ \sum_{k=0}^{K-1} c^k a(\omega_k'') + \sum_{k=K}^{\infty} c^k a(\omega_k^*) : \omega_k'' \in E_k^N, k = 0, 1, 2, \ldots, K-1 \right\}.$$

Then

$$\#(F_k^N) \le \prod_{k=0}^{K-1} \#(E_k^N). \tag{24}$$

Now we prove F_k^N is a $(\frac{\varepsilon}{3(1-c)}, N)$-spanning subset in $\sigma^{-N}(x)$. Indeed, for any $x' = \sum_{k=0}^{\infty} c^k a(\omega'_k) \in \sigma^{-N}(x)$, one has that $\omega'_k \in T^{-N}(\omega_k)$ for any $k \in \mathbb{Z}_+$. Thus, for any $k = 0, 1, \ldots, K-1$, there exists $\omega''_k \in E_k^N$ such that $\rho_N(\omega'_k, \omega''_k) < \delta$, which together with (23), implies that $d_N(a(\omega'_k), a(\omega''_k)) < \varepsilon/6$. Therefore,

$$
\begin{aligned}
& d_N\left(\sum_{k=0}^{\infty} c^k a(\omega'_k), \sum_{k=0}^{K-1} c^k a(\omega''_k) + \sum_{k=K}^{\infty} c^k a(\omega^*_k)\right) \\
&= \max_{0 \le l < N} \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \sum_{k=0}^{K-1} c^k \left(a(T^l(\omega'_k)) - a(T^l(\omega''_k))\right) + \sum_{k=K}^{\infty} c^k \left(a(T^l(\omega'_k)) - a(T^l \omega^*_k)\right) \right| \\
&\le \sum_{k=0}^{K-1} c^k \max_{0 \le l < N} \sum_{n=1}^{\infty} \frac{1}{2^n} |a(T^l(\omega'_k)), a(T^l(\omega''_k))| + M \cdot \sum_{k=K}^{\infty} c^k \\
&\le \sum_{k=0}^{K-1} c^k d_N(a(\omega'_k), a(\omega''_k)) + \frac{\varepsilon}{6(1-c)} \\
&\le \left(\sum_{k=0}^{\infty} c^k\right) \cdot \frac{\varepsilon}{6} + \frac{\varepsilon}{6(1-c)} = \frac{\varepsilon}{3(1-c)},
\end{aligned}
$$

which shows that the set F_k^N is a $(N, \frac{\varepsilon}{3(1-c)})$-spanning subset of $\sigma^{-N}(x)$. Since $\varepsilon/6 \le Mc^{K-1}$, we have

$$
\log \frac{6M}{c\varepsilon} \ge K \log \frac{1}{c},
$$

which together with (24), implies that for each $N \in \mathbb{Z}_+$,

$$
\frac{\log \sup_{x \in X} r(N, \frac{\varepsilon}{3(1-c)}, \sigma^{-N}(x))}{N \log(\frac{6M}{c\varepsilon})} \le \frac{\log \sup_{\omega \in \Omega} r(N, \delta, T^{-N}(\omega))}{N \log(1/c)}.
$$

Letting $N \to \infty$, and then letting $\delta \to 0$, we have that

$$
\limsup_{N \to \infty} \frac{\log \sup_{x \in X} r(N, \frac{\varepsilon}{3(1-c)}, \sigma^{-N}(x))}{N \log(\frac{6M}{c\varepsilon})} \le \frac{h_{\mathrm{m}}(T, \Omega)}{\log(1/c)}.
$$

Let $\varepsilon \to 0$. Then we deduce that

$$
\overline{\mathrm{mdim}}_{\mathrm{M,m}}(X, \sigma, d) \le \frac{h_{\mathrm{m}}(T, \Omega)}{\log(1/c)}.
$$

The proof is completed. □

8.3. **Zipper map.** In this subsection, we consider the preimage metric mean dimension of zipper maps. Firstly, we introduce the family of maps, which is used to define the zipper map. Their graph will be a zipper curve. As functions, they were introduced by Bruneau [3] as extremal points in certain functional spaces.

Denote by $\mathscr{C}$ the Banach space of continuous functions defined on $[0,1]$ with values in $\mathbb{R}$ endowed with its usual norm $\|f\| = \sup_{x \in [0,1]} |f(x)|$, and by $\mathscr{C}_0$ the convex, closed subset of the functions f with range $[0,1]$ and such that $f(0) = 0$ and $f(1) = 1$.

Let $p = ((x_1, y_1), (x_2, y_2)) \in (0,1)^2$ be a pair of points in the unit square such that $x_2 > x_1$ and $y_2 < y_1$, and let $\Phi_p : \mathscr{C}_0 \to \mathscr{C}_0$ be the map defined by

$$
\Phi_p f(x) = \begin{cases} y_1 f(\frac{x}{x_1}) & \text{if } x \in [0, x_1] \\ y_1 - (y_1 - y_2) f(\frac{x - x_1}{x_2 - x_1}) & \text{if } x \in [x_1, x_2] \\ y_2 + (1 - y_2) f(\frac{x - x_2}{1 - x_2}) & \text{if } x \in [x_2, 1]. \end{cases}
$$

Then Φ_p is a contraction in the uniform norm, and thus has a unique fixed point $Z_p \in \mathscr{C}_0$. This map Z_p is said to be the zipper map of parameter p. In [14], authors proved a class of zipper maps have positive metric mean dimension relative to the Euclidean metric d. However, we prove that the preimage mean metric dimension of each zipper map is zero.

Theorem 8.4. *Let Z_p be a zipper map. Then*

$$\mathrm{mdim}_{\mathrm{M,m}}\Big([0,1], Z_p, d\Big) = 0.$$

Proof. We define intervals

$$I_0 = [0, x_1],\ I_1 = [x_1, x_2],\ I_2 = [x_2, 1]$$

and

$$J_0 = [0, y_1],\ J_1 = [y_2, y_1],\ J_2 = [y_2, 1].$$

Since Z_p is the fixed point of Φ_p, it follows that

$$Z_p(I_i) = J_i,\ \text{for any } i = 0, 1, 2. \tag{25}$$

Note that for any $x \in J_0$, $\#(Z_p^{-1}(x)) = 1$; for any $x \in J_1$, $\#(Z_p^{-1}(x)) = 3$; for any $x \in J_2$, $\#(Z_p^{-1}(x)) = 1$. Thus, given any $x \in [0,1]$, for any $\varepsilon > 0$,

$$s(n, \varepsilon, f^{-n}(x)) \leq 3^n \text{ for any } n \in \mathbb{N}.$$

This shows that $\mathrm{mdim}_{\mathrm{M,m}}\Big([0,1], Z_p, d\Big) = 0$. ☐

9. Final comments

In [1] the author proved, among many interesting results, that in a Riemannian manifold of dimension $n \geq 2$, the set of continuous self-maps with upper metric mean dimension equal to n is a C^0-residual in the space of continuous self-maps. We believe the arguments presented there can pave the way for us to prove the following conjecture:

Conjecture. In a Riemannian manifold of dimension $n \geq 2$, the set of continuous self-maps with upper preimage metric mean dimension equal to n is a C^0-residual in the space of continuous self-maps.

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