GENERIC SHARP SHIFT EMBEDDING FOR AMENABLE GROUP ACTIONS WITH THE UNIFORM ROKHLIN PROPERTY AND COMPARISON

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Abstract. Naryshkin recently proved that a free action of a countably infinite discrete amenable group G on a compact metrizable space X with the uniform Rokhlin property and comparison (URPC) admits an equivariant embedding into $([0,1]^m)^G$ whenever $\mathrm{mdim}(X,G) < m/2$. We strengthen this result from existence to genericity by a substantially different method: the set of observables $f \in C(X,[0,1]^m)$ whose orbit maps are equivariant embeddings into $([0,1]^m)^G$ is a dense G_δ subset of $C(X,[0,1]^m)$. The proof combines an adaptation of the Kakutani skyscraper construction used by Gutman–Tsukamoto with a new polynomial perturbation method, departing from the predominantly linear perturbation approaches used in earlier equivariant embedding literature.

1. Introduction

A fundamental research endeavor in the theory of topological dynamical systems originated in Auslander's famous 1988 question [Aus88] : *Does every minimal homeomorphism of a compact metrizable space admit an equivariant embedding into the shift on* $[0,1]^{\mathbb{Z}}$*?* Mean dimension supplied the decisive obstruction. Introduced by Gromov [Gro99] in 1999 and developed systematically by Lindenstrauss and Weiss [LW00], it measures the average amount of topological dimension per iterate. The shift on $([0,1]^m)^{\mathbb{Z}}$ has mean dimension m, and mean dimension is monotone under equivariant embeddings. Lindenstrauss and Weiss constructed a minimal system of mean dimension greater than one, thereby answering Auslander's question negatively. Lindenstrauss then established a striking partial converse: a minimal system with $\mathrm{mdim}(X,T) < \frac{m}{36}$ *generically*[1] embeds equivariantly into the m-cubical shift

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[1]Genericity means that the set of observables $f \in C(X,[0,1]^m)$ for which the orbit map $x \mapsto (f(T^n x))_{n\in\mathbb{Z}}$ is an embedding is a dense G_δ subset of $C(X,[0,1]^m)$, equipped with the uniform topology.

[Lin99]. In the same paper, he asked for the largest constant c for which the bound $\mathrm{mdim}(X,T) < cm$ guarantees an equivariant embedding. Gutman and Tsukamoto [GT20] solved the problem by proving the existence[2] of an equivariant embedding for any $c < \frac{1}{2}$. As Lindenstrauss and Tsukamoto [LT14] had previously constructed minimal systems of mean dimension exactly $m/2$ not embeddable into the m-cubical shift, the constant $1/2$, and the strict inequality, were shown to be optimal.

The marker property introduced in [Gut15] based on related notions of [Dow06] and [Kri82] is a natural generalization of minimality in the context of the embedding problem. For a group action $G \curvearrowright X$, it requires that for every finite $F \subset G$ there be an open set U whose translates gU, $g \in F$, are pairwise disjoint and whose full family of translates covers X. Every free minimal action has this property. Following the development of dynamical tiling methods in [GLT16], Gutman, Qiao, and Tsukamoto proved the sharp embedding theorem for $\mathbb{Z}^k$-actions with the marker property [GQT19].

In view of the fundamental role of the Rokhlin lemma in coding arguments in ergodic theory ([Kri70, Orn70, Orn74]), the uniform Rokhlin property (URP) introduced by Niu [Niu22] in his study of crossed-product C^*-algebras associated with free minimal actions of amenable groups[3], offers a particularly natural setting for the embedding problem[4]. It stipulates the existence of finite families of towers with arbitrarily invariant shapes and mutually disjoint levels, whose union covers all but an arbitrarily small part of the space, uniformly over all invariant probability measures. Naryshkin proved that a free action of a countably infinite discrete amenable group G with URP and *comparison*[5] (URPC), satisfying $\mathrm{mdim}(X,G) < m/2$, embeds into $([0,1]^m)^G$ [Nar24, Theorem A]. Note that for free abelian-group actions, URPC is equivalent to the marker property ([Nar24, Corollary E]). However, the question whether general amenable free minimal actions satisfy URPC remains open.[6]

In his proof of the embedding theorem, Naryshkin used an ingenious construction that encodes an entire sequence of URPC covers in a single continuous observable [Nar24, Lemmas 6.6–6.7]. The Baire-category argument is then carried out in a closed proper subclass of observables constrained by that function. The passage from existence to genericity requires one to

[2]Unlike the result of [Lin99], the result in [GT20] is an existence result, not establishing genericity. The generic embedding result for minimal $\mathbb{Z}$-actions was first proven in [Lev23].

[3]This is one of many manifestations of the interaction between mean dimension theory and the structure and classification of C^*-algebras; see, for example, [Ker20, KS20, Niu22] and the discussion in [Nar24].

[4]There have, however, also been deep recent developments for systems beyond this class; see [DL25, Mey26, Shi26].

[5]Essentially, comparison allows sets that are smaller in every invariant measure to be moved piece by piece into larger ones by group translates, see Definition 2.3.

[6]Very recently, it was shown in [GL26] that every free action of a countably infinite amenable group on a finite-dimensional compact metrizable space has URPC.

perform the Baire-category argument in the *entire* observable space. For $f \in C(X,[0,1]^m)$, denote its *orbit map* by

$$f^G : X \to ([0,1]^m)^G, \qquad f^G(x)_g = f(gx), \quad g \in G.$$

We state our main theorem.

Theorem A. *Let a countable*[7] *discrete amenable group G act freely by homeomorphisms on a nonempty compact metrizable space X, and assume that the action has URPC. If* $\operatorname{mdim}(X,G) < \frac{m}{2}$, *then*

$$\mathcal{E}_m(X,G) := \{ f \in C(X,[0,1]^m) : f^G \text{ is a topological embedding} \}$$

is a dense G_δ subset of $C(X,[0,1]^m)$.

In particular, the conclusion applies to free abelian-group actions with the marker property, and hence to free minimal abelian-group actions, under the stated mean-dimension bound. Sharpness as a uniform theorem already follows from the minimal $\mathbb{Z}$-examples of [LT14]. This also gives a dynamical counterpart of the generic Menger–Nöbeling theorem: for a compact metrizable space K with $\dim K < m/2$, embeddings into $[0,1]^m$ form a dense G_δ subset of $C(K,[0,1]^m)$ [HW41, Eng78].

Our proof differs significantly from Naryshkin's proof. Ultimately generic embedding is achieved through a novel method detailed in Section 5. The key ingredient, proven in that section, is Proposition 4.2, which we call *finite-window localization up to a buffer*. Its role can be described as follows. From a *URPC witness* we construct finitely many compact sets K_ℓ with open neighborhoods O_ℓ. Suitable translates of the K_ℓ cover X, while the corresponding translates of the O_ℓ are contained in tower levels or open sets covering the remainder.

Starting from a suitable partition-of-unity approximation f_* of an arbitrary observable, we aim to perturb f_* arbitrarily slightly to q so that, for a finite set $F \subset G$,

$$q^F(K_\ell) \cap q^F(X \setminus O_\ell) = \emptyset, \qquad q^F(x) := \big(q(gx)\big)_{g\in F}. \tag{1.1}$$

Thus, whenever $x \in K_\ell$, any point with the same observations as x on the finite window F must belong to O_ℓ. Once this localization has been achieved, applying it to suitable translates shows that points with identical full orbit observations lie in a common tower level or in a common member of the open cover of the remainder. This allows the subsequent separation argument to be carried out within these regions. Indeed, we use general-position perturbations of orbit blocks, as in the *Kakutani skyscraper decomposition* technique of Gutman–Tsukamoto [GT14], and combine the resulting block maps into a single continuous observable by placing their coordinates along the corresponding tower levels. The buffer between K_ℓ and $X \setminus O_\ell$ makes the localization stable under the additional (sufficiently small) perturbations.

[7] If G is finite, the conclusion holds assuming only freeness by [GLM25, Theorem 1.1]. Henceforth, we assume that G is infinite.

The main challenge is to achieve (1.1) by an arbitrarily small perturbation. Departing from the predominantly linear perturbation approaches used in earlier equivariant embedding arguments, we introduce *polynomial perturbations of partition-of-unity maps*. Indeed, if $f_* = \sum_j v_j \psi_j$, where $v_j \in (0,1)^m$ and $\boldsymbol{\psi} = (\psi_1, \dots, \psi_J)$ is a partition of unity, the perturbations have the form

$$q_\Theta(z)_i = f_*(z)_i + \langle \Xi(z), \theta^{(i)} \rangle, \qquad 1 \le i \le m,$$

where Ξ consists of translated cutoffs and monomials of bounded degree in finitely many translates of $\boldsymbol{\psi}$.[8] The coefficients Θ may be chosen arbitrarily close to zero. The proof is completed by a combination of combinatorial rank estimates and semialgebraic dimension arguments.

Organization of the paper. Section 2 introduces notation and background material used throughout the paper. In Section 3, we develop the low-dimensional coordinate systems needed for the perturbation argument. Section 4 establishes the finite-scale approximation theorem and derives Theorem A by a Baire-category argument, assuming the *finite-window localization up to a buffer* result. Finally, Section 5 proves this localization result and completes the proof.

2. Preliminaries

Throughout the paper, G is a countably infinite discrete amenable group acting on the left by homeomorphisms on a nonempty compact metrizable space X, and e denotes the identity of G. We fix $m \in \mathbb{N}$ and a compatible metric ρ on X. For subsets $A, B \subset X$, we write $A \Subset B$ if $\overline{A} \subset B$. All inclusions are understood to allow equality.

We write $\mathcal{P}_G(X)$ for the space of G-invariant Borel probability measures on X. We equip $\mathbb{R}^m$ and its finite products with the maximum norm, denoted by $\|\cdot\|$, and function spaces with the corresponding uniform norm $\|\cdot\|_\infty$.

2.1. **Basic notation.** For $q \in C(X, [0,1]^m)$ and any subset $D \subset G$, finite or infinite, write $q^D(x) := (q(gx))_{g \in D}$. Every finite product of Euclidean cubes is equipped with the maximum norm. We equip $([0,1]^m)^G$ with the left shift $(h \cdot z)_g = z_{gh}$, $g, h \in G$, and define the equivariant orbit map

$$q^G : X \longrightarrow ([0,1]^m)^G, \qquad q^G(x)_g = q(gx).$$

Then $q^G(hx) = h \cdot q^G(x)$.

More generally, if $\xi = (\xi_\lambda)_{\lambda \in L}$ is a finite tuple of continuous functions and $D \subset G$ is finite, put

$$\xi^D(x) := \big(\xi_\lambda(dx)\big)_{(d,\lambda) \in D \times L}.$$

For a single vector-valued map this reduces to $\xi^D(x) = (\xi(dx))_{d \in D}$.

[8] Polynomial perturbations have been used in a non-equivariant Euclidean-space setting in the context of the *Takens-type delay embedding theorems*, e.g. see [SYC91, BGŚ24].

For a finite indexed family $\mathcal{C} = (C_j)_{j=1}^J$ of subsets of a space X, define

$$\mathrm{loc}_{\mathcal{C}}(x) := \{j : x \in C_j\}, \qquad \mathrm{mult}_{\mathcal{C}}(x) := |\,\mathrm{loc}_{\mathcal{C}}(x)|.$$

The multiplicity of $\mathcal{C}$ is

$$\mathrm{mult}(\mathcal{C}) := \sup_{x\in X} \mathrm{mult}_{\mathcal{C}}(x).$$

If $\mathcal{C}$ is a finite cover, define

$$\mathrm{ord}_{\mathcal{C}}(x) := \mathrm{mult}_{\mathcal{C}}(x) - 1, \qquad \mathrm{ord}(\mathcal{C}) := \mathrm{mult}(\mathcal{C}) - 1 = \sup_{x\in X} \mathrm{ord}_{\mathcal{C}}(x).$$

For a finite open cover $\mathcal{U}$ of a compact metrizable space, set

$$D(\mathcal{U}) := \min\bigl\{\mathrm{ord}(\mathcal{V}) : \mathcal{V} \text{ is a finite open cover refining } \mathcal{U}\bigr\}.$$

For a cover $\mathcal{U}$ of (X, ρ), write

$$\mathrm{mesh}(\mathcal{U}) := \sup_{U\in\mathcal{U}} \mathrm{diam}_\rho(U).$$

A map $H : (X, \rho) \to Y$ is an η-**embedding** if $H(x) = H(y) \Longrightarrow \rho(x, y) < \eta$.

Let $H : Y \to Z$ be a map and $C \subset Y$. We say that H **encodes** C if

$$H^{-1}(H(C)) = C.$$

We say that H **encodes a finite family** if it encodes every member of that family.

2.2. Partitions of unity and simplicial notation.

Let

$$\boldsymbol{\psi} = (\psi_1, \ldots, \psi_J) : X \to \Delta^{J-1} := \left\{ (z_1, \ldots, z_J) \in [0, 1]^J : \sum_{j=1}^{J} z_j = 1 \right\}$$

be a partition of unity. For a finite set $E \subset G$, denote

$$\boldsymbol{\psi}^E(x) := (\boldsymbol{\psi}(tx))_{t\in E}.$$

Given an indexed open cover $(\widetilde{O_j})_{j=1}^J$, we say that $\boldsymbol{\psi}$ is **honest** for $(\widetilde{O_j})_{j=1}^J$ if $\{\psi_j > 0\} = \widetilde{O_j}$, $1 \le j \le J$. Suppose moreover that a closed cover $(D_j)_{j=1}^J$ is fixed and $\mathrm{supp}\,\psi_j \subset D_j$, $1 \le j \le J$. We call the triple

$$\mathcal{H} := \bigl((\widetilde{O_j})_{j=1}^J, (D_j)_{j=1}^J, \boldsymbol{\psi}\bigr)$$

an **honest partition structure**.

For a nonempty $I \subset \{1, \ldots, J\}$, let Δ_I be the coordinate face of Δ^{J-1} spanned by the vertices indexed by I. Write $\mathcal{D} := (D_j)_{j=1}^J$. Since $\mathrm{supp}\,\psi_j \subset D_j$,

$$\boldsymbol{\psi}(x) \in \Delta_{\mathrm{loc}_{\mathcal{D}}(x)}, \qquad \dim \Delta_{\mathrm{loc}_{\mathcal{D}}(x)} = \mathrm{ord}_{\mathcal{D}}(x).$$

For later use, we also fix the polynomial coordinates on the simplex. For $d \in \mathbb{N}_0$, let

$$\mathcal{A}_{J,d} := \bigl\{\alpha = (\alpha_1, \ldots, \alpha_J) \in \mathbb{N}_0^J : |\alpha| := \alpha_1 + \cdots + \alpha_J \le d\bigr\},$$

and denote $M_{J,d} := |\mathcal{A}_{J,d}| = \binom{J+d}{d}$. For $z = (z_1, \dots, z_J) \in \Delta^{J-1}$ and $\alpha \in \mathcal{A}_{J,d}$, write $z^\alpha := z_1^{\alpha_1} \cdots z_J^{\alpha_J}$. We define the **degree-d Veronese map** by

$$\Phi_d : \Delta^{J-1} \longrightarrow \mathbb{R}^{M_{J,d}}, \qquad \Phi_d(z) := (z^\alpha)_{\alpha \in \mathcal{A}_{J,d}}.$$

Thus every polynomial P on $\mathbb{R}^J$ of degree at most d can be written in the form

$$P(z) = \langle a_P, \Phi_d(z) \rangle \qquad \text{for some } a_P \in \mathbb{R}^{M_{J,d}}. \tag{2.1}$$

We shall also use the following standard interpolation terminology.

Definition 2.1. Let Y be a set and let $k \geq 1$. A finite family of functions $h_1, \dots, h_M : Y \to \mathbb{R}$ is called **k-interpolating on** Y if, for every $1 \leq q \leq k$, every collection of pairwise distinct points $z_1, \dots, z_q \in Y$, and every $(u_1, \dots, u_q) \in \mathbb{R}^q$, there exists $\theta = (\theta_1, \dots, \theta_M) \in \mathbb{R}^M$ such that

$$\sum_{\alpha=1}^{M} \theta_\alpha h_\alpha(z_i) = u_i, \qquad 1 \leq i \leq q.$$

By the classical polynomial interpolation argument [SYC91, Lemma 4.1(1)]; see also [GS00, Section 1.2, eq. (1.9)], and (2.1),

$$\text{the family } \{z \mapsto z^\alpha : \alpha \in \mathcal{A}_{J,d}\} \text{ is } (d+1)\text{-interpolating on } \Delta^{J-1}. \tag{2.2}$$

2.3. **Uniform Rokhlin property and comparison.** An **open castle** is a finite family $\{(S_i, V_i)\}_{i=1}^r$, where each $S_i \subset G$ is finite and each $V_i \subset X$ is open, such that the levels sV_i, $1 \leq i \leq r, s \in S_i$, are pairwise disjoint. Each pair (S_i, V_i) is a **tower**, S_i is its **shape**, V_i is its **base**, and the sets sV_i, $s \in S_i$, are its **levels**. The closed set $X \setminus \bigcup_{i=1}^r S_i V_i$ is the **remainder** of the castle.

For a finite set $K \subset G$ and $\alpha > 0$, a nonempty finite set $S \subset G$ is called (K, α)-**invariant** if $\left|\bigcap_{k \in K} kS\right| > (1 - \alpha)|S|$.

For a finite set $L \subset G$, an open set $U \subset X$ is L-**free** if the family $\{\ell U : \ell \in L\}$ is pairwise disjoint. It is a **sweep-out set** if $MU = X$ for some finite $M \subset G$. The action has the **marker property** if it admits an L-free sweep-out set for every finite $L \subset G$.

The following property of finite invariance is a standard consequence; see, for example, [KL16, Section 4.1].

Lemma 2.2. *Let $A, E, K \subset G$ be finite, with $e \in A$ and $E \neq \emptyset$, and let $\varepsilon_A, \varepsilon_E > 0$ and $\delta \in (0, 1)$. Then there exist a finite set $K_0 \subset G$ and $\delta_0 \in (0, 1)$ such that every nonempty finite (K_0, δ_0)-invariant set $S \subset G$ satisfies $|AS| < (1 + \varepsilon_A)|S|$ and $|ES| < (1 + \varepsilon_E)|S|$, and AS is (K, δ)-invariant.*

For a closed set $A \subset X$ and an open set $B \subset X$, we write $A \prec B$ if there exist a finite open cover $\mathcal{U}$ of A and elements $g_U \in G$, $U \in \mathcal{U}$, such that the sets $\{g_U U : U \in \mathcal{U}\}$ are pairwise disjoint subsets of B.

Definition 2.3 (Kerr [Ker20, Definition 3.2])**.** The action $G \curvearrowright X$ has **comparison** if $A \prec B$ whenever $A \subset X$ is closed, $B \subset X$ is open, and $\mu(A) < \mu(B)$ for every $\mu \in \mathcal{P}_G(X)$.

Definition 2.4 (Niu [Niu22])**.** The action $G \curvearrowright X$ has the **uniform Rokhlin property** (URP) if, for every finite $K \subset G$ and every $\varepsilon > 0$, there is an open castle $\{(S_i, V_i)\}_{i=1}^n$ such that every S_i is (K, ε)-invariant and

$$\sup_{\mu \in \mathcal{P}_G(X)} \mu\left(X \setminus \bigsqcup_{i=1}^{n} S_i V_i\right) < \varepsilon.$$

As the remainder is closed, this invariant-measure formulation agrees with Naryshkin's upper-density formulation; see [Nar24, Subsection 2.1].

Remark 2.5. If an action has URP, then it is free almost everywhere for every invariant probability measure. This assertion does not require topological freeness. Write $\mathrm{Fix}(g) := \{x \in X : gx = x\}$. Fix $g \neq e$ and $\varepsilon > 0$, and apply URP with $K = \{e, g^{-1}\}$. For each resulting shape S_i, denote $C_i := S_i \cap g^{-1} S_i$. Then $|S_i \setminus C_i| < \varepsilon |S_i|$. If $s \in C_i$ and $z \in sV_i$ satisfies $gz = z$, then $z = gz \in gsV_i$, contradicting the disjointness of the distinct levels sV_i and gsV_i. Thus, for every $\mu \in \mathcal{P}_G(X)$,

$$\begin{aligned} \mu(\mathrm{Fix}(g)) &\leq \mu\left(X \setminus \bigsqcup_i S_i V_i\right) + \sum_i |S_i \setminus C_i| \mu(V_i) \\ &< \varepsilon + \varepsilon \sum_i |S_i| \mu(V_i) \leq 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \downarrow 0$ gives $\mu(\mathrm{Fix}(g)) = 0$. Since G is countable, the action is free μ-almost everywhere for every invariant μ.

Definition 2.6 (Naryshkin [Nar24])**.** The action $G \curvearrowright X$ has **URPC** if it has both the uniform Rokhlin property and comparison.

We shall use the following finite-data formulation.

Definition 2.7 (Naryshkin [Nar24, Definition 3.24])**.** Let $K \subset G$ be finite and let $\alpha > 0$. A (K, α)-**URPC witness** consists of finite data

$$\mathfrak{W} = \left(\{S_i\}_{i=1}^r, \{V_i\}_{i=1}^r, \{g_j\}_{j=1}^q, \{U_j\}_{j=1}^q\right)$$

such that:

(U1) $\{(S_i, V_i)\}_{i=1}^r$ is an open castle such that every S_i is (K, α)-invariant;
(U2) the open sets $U_1, \ldots, U_q$ cover the castle remainder $X \setminus \bigsqcup_{i=1}^r S_i V_i$;
(U3) the index set $\{1, \ldots, q\}$ is partitioned into subsets $J_{i,c}$, where $1 \leq i \leq r$, and $1 \leq c \leq \lfloor \alpha |S_i| \rfloor$, such that, whenever $j \in J_{i,c}$, $g_j U_j \subset V_i$, and, for every fixed pair (i, c), the sets $\{g_j U_j : j \in J_{i,c}\}$ are pairwise disjoint.

Empty index ranges and empty sets $J_{i,c}$ are allowed. The elements g_j are called **transporters**, and the integers c are called **colors**.

Proposition 2.8 (Naryshkin [Nar24, Theorem 3.20(iii)])**.** *A free action has URPC if and only if it admits a (K, α)-URPC witness for every finite $K \subset G$ and every $\alpha > 0$.*

2.4. **Amenability and mean dimension.** A sequence $(F_n)_{n\geq 1}$ of nonempty finite subsets of G is a **Følner sequence** if

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0 \qquad \text{for every } g \in G.$$

Since G is amenable, such sequences exist.

For a finite open cover $\mathcal{U}$ of X and a nonempty finite set $F \subset G$, denote

$$\mathcal{U}^F := \bigvee_{g\in F} g^{-1}\mathcal{U} = \left\{ \bigcap_{g\in F} g^{-1}U_g : U_g \in \mathcal{U} \right\}.$$

Let $(F_n)_{n\geq 1}$ be a Følner sequence. The mean dimension of the cover $\mathcal{U}$ is

$$\operatorname{mdim}(\mathcal{U}, G) := \lim_{n\to\infty} \frac{D(\mathcal{U}^{F_n})}{|F_n|}.$$

By the Ornstein–Weiss lemma [OW87, LW00], the limit exists and is independent of the choice of the Følner sequence. The **mean dimension** of the action is

$$\operatorname{mdim}(X, G) := \sup_{\mathcal{U}} \operatorname{mdim}(\mathcal{U}, G),$$

where the supremum is taken over all finite open covers $\mathcal{U}$ of X.

Let $\mathcal{U}$ and $\mathcal{V}$ be covers of a topological space X. We say that $\mathcal{V}$ **refines** $\mathcal{U}$, and write $\mathcal{V} \succ \mathcal{U}$, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$.

Meyerovitch [Mey26] defined the following invariant, called the **dynamical dimension**:

$$\operatorname{ddim}(X, G) := \sup_{\mathcal{A}} \inf_{\mathcal{B}\succ\mathcal{A}} \sup_{\mu\in\mathcal{P}_G(X)} \int_X \operatorname{ord}_{\mathcal{B}}(x)\, d\mu(x),$$

where $\mathcal{A}$ and $\mathcal{B}$ range over finite open covers of X. By [Mey26, Theorem 9.2], URP implies

$$\operatorname{ddim}(X, G) = \operatorname{mdim}(X, G). \tag{2.3}$$

2.5. **Semialgebraic sets.** For a topological space Z, $\dim Z$ denotes its Lebesgue covering dimension.

We recall facts about semialgebraic sets that will be used below; see [BCR98, Sections 2.2 and 2.8].

A subset $Z \subset \mathbb{R}^N$ is called **semialgebraic** if it is a finite union of sets of the form

$$\left\{ x \in \mathbb{R}^N : p_1(x) = \cdots = p_r(x) = 0,\ q_1(x) > 0, \ldots, q_s(x) > 0 \right\},$$

where $p_1, \ldots, p_r, q_1, \ldots, q_s$ are real polynomials.

If $Z \subset \mathbb{R}^N$ is semialgebraic, a map $f : Z \to \mathbb{R}^M$ is called **semialgebraic** if its graph

$$\Gamma_f := \{(z, f(z)) : z \in Z\} \subset \mathbb{R}^{N+M}$$

is semialgebraic. A map with values in a finite-dimensional vector space, or in a matrix space, is semialgebraic if its coordinate functions are semialgebraic.

Lemma 2.9. *Let X, Y be semialgebraic sets.*

(i) *If $X = X_1 \cup \cdots \cup X_N$ with each X_i semialgebraic, then*

$$\dim X = \max_{1 \le i \le N} \dim X_i.$$

(ii) *The product $X \times Y$ is semialgebraic and* $\dim(X \times Y) = \dim X + \dim Y$.

(iii) *The closure $\overline{X}$ is semialgebraic and* $\dim \overline{X} = \dim X$.

(iv) *If $f : X \to Y$ is semialgebraic, $A \subset X$ is semialgebraic, and $B \subset Y$ is semialgebraic, then $f(A)$ and $f^{-1}(B)$ are semialgebraic and*

$$\dim f(A) \le \dim A.$$

(v) *Let $f : X \to Y$ be a continuous semialgebraic map. If every nonempty fibre $f^{-1}(y)$ has dimension at most q, then*

$$\dim X \le \dim f(X) + q \le \dim Y + q.$$

(vi) *If $X \subset \mathbb{R}^N$ and* $\dim X < N$, *then*

$$\operatorname{int}_{\mathbb{R}^N}(\overline{X}) = \emptyset.$$

That is, X has nowhere dense closure.

Proof. (i) and (ii) are [BCR98, Proposition 2.8.5 (i) (ii)], respectively.

For (iii), the semialgebraicity of $\overline{X}$ follows from [BCR98, Proposition 2.2.2], while $\dim \overline{X} = \dim X$ is [BCR98, Proposition 2.8.2].

For (iv), the semialgebraicity of $f(A)$ and $f^{-1}(B)$ follows from [BCR98, Proposition 2.2.7], and $\dim f(A) \le \dim A$ is [BCR98, Theorem 2.8.8].

For (v), apply Hardt's semialgebraic triviality theorem [BCR98, Theorem 9.3.2]. There is a finite semialgebraic partition $f(X) = \bigsqcup_{\ell=1}^{s} Y_\ell$ such that, for each ℓ, there is a semialgebraic homeomorphism $h_\ell : f^{-1}(Y_\ell) \to Y_\ell \times F_\ell$ such that $\mathrm{pr}_1 \circ h_\ell = f|_{f^{-1}(Y_\ell)}$. Here pr_1 is projection onto Y_ℓ. In particular, h_ℓ identifies each fibre over $y \in Y_\ell$ with $\{y\} \times F_\ell$. Hence $\dim F_\ell \le q$. Using (i) and (ii), we obtain

$$\begin{aligned}\dim X &= \max_{1 \le \ell \le s} \dim f^{-1}(Y_\ell) = \max_{1 \le \ell \le s} \big(\dim Y_\ell + \dim F_\ell\big) \\ &\le \max_{1 \le \ell \le s} \dim Y_\ell + q = \dim f(X) + q \le \dim Y + q.\end{aligned}$$

Finally, suppose that $X \subset \mathbb{R}^N$ and $\dim X < N$. By (iii), $\dim \overline{X} = \dim X < N$. If $\operatorname{int}_{\mathbb{R}^N}(\overline{X})$ were nonempty, then it would be a nonempty open semialgebraic subset of $\mathbb{R}^N$, and therefore

$$\dim \operatorname{int}_{\mathbb{R}^N}(\overline{X}) = N$$

by [BCR98, Proposition 2.8.4]. Since $\mathrm{int}_{\mathbb{R}^N}(\overline{X}) \subset \overline{X}$, this would imply $N \le \dim \overline{X}$, a contradiction. Thus

$$\mathrm{int}_{\mathbb{R}^N}(\overline{X}) = \emptyset.$$

Since $\overline{X}$ is closed, it follows that X has nowhere dense closure. □

Proposition 2.10. *Let Z be a semialgebraic set and let $M : Z \longrightarrow \mathrm{Mat}_{n\times P}(\mathbb{R})$ be a semialgebraic map. For every integer $r \ge 0$, the set $Z_r := \{z \in Z : \mathrm{rank}\, M(z) = r\}$ is semialgebraic. The nonempty sets Z_r form a finite partition of Z.*

Proof. Note that $\mathrm{rank}\, M(z) \le r$ is equivalent to the vanishing of all $(r+1)\times(r+1)$ minors, while $\mathrm{rank}\, M(z) \ge r$ is equivalent to the nonvanishing of some $r \times r$ minor. These are semialgebraic conditions. Only the ranks $0, \dots, \min\{n, P\}$ can occur. See also [BCR98, Sections 2.1–2.3]. □

3. Global low-dimensional coordinates

Assuming URP, we use the equality of dynamical dimension and mean dimension to construct a partition of unity with uniformly controlled local order along sufficiently invariant finite orbit windows. The resulting orbit names lie in finite unions of products of simplex faces with the corresponding dimension bound.

Lemma 3.1. *Assume that $G \curvearrowright X$ has URP and that $\mathrm{mdim}(X,G) < a$. Let $\mathcal{W}$ be a finite open cover of X, let $f_0 \in C(X,(0,1)^m)$, and let $\varepsilon > 0$. Then there exist an honest partition structure $\mathcal{H} = \big((\widetilde{O}_j)_{j=1}^J, (D_j)_{j=1}^J, \boldsymbol{\psi}\big)$ and vectors $v_1, \dots, v_J \in (0,1)^m$ such that*

- (i) *the closed cover $\mathcal{D} = (D_j)_{j=1}^J$ refines $\mathcal{W}$;*
- (ii) *the affine observable $f_*(x) := \sum_{j=1}^J v_j \psi_j(x)$ satisfies $\|f_* - f_0\|_\infty < \varepsilon$;*
- (iii) $\sup_{\mu\in\mathcal{P}_G(X)} \int_X \mathrm{ord}_{\mathcal{D}}(x)\, d\mu(x) < a$;
- (iv) *if $\boldsymbol{\psi}(x) = \boldsymbol{\psi}(y)$, then x and y lie in a common member of $\mathcal{W}$.*

Proof. By (2.3), URP implies $\mathrm{ddim}(X,G) = \mathrm{mdim}(X,G)$. Choose $\mathrm{ddim}(X,G) < a_0 < a$.

By uniform continuity of f_0, there is a finite open cover $\mathcal{A}$ of X which refines $\mathcal{W}$ and such that $\|f_0(x) - f_0(y)\|_\infty < \varepsilon$ whenever x and y belong to a common member of $\mathcal{A}$.

By the definition of $\mathrm{ddim}(X,G)$, there is a finite open refinement $\mathcal{B} = (B_j)_{j=1}^J$ of $\mathcal{A}$ such that

$$\sup_{\mu\in\mathcal{P}_G(X)} \int_X \mathrm{ord}_{\mathcal{B}}(x) d\mu(x) < a_0.$$

Discarding empty members, we may assume that every B_j is nonempty. For each j, choose $A_j \in \mathcal{A}$ such that $B_j \subset A_j$.

Choose a closed shrinking $(C_j)_{j=1}^J$ of $(B_j)_{j=1}^J$ such that

$$C_j \subset B_j \qquad \text{and} \qquad X = \bigcup_{j=1}^{J} C_j.$$

Choose open sets $\widetilde{O_j}$ satisfying $C_j \subset \widetilde{O_j} \Subset B_j$. The sets $\widetilde{O_j}$ still cover X. Set $D_j := \overline{\widetilde{O_j}}$. Then $(D_j)_{j=1}^J$ is a finite closed cover of X, and $D_j \subset B_j \subset A_j$. Since $\mathcal{A}$ refines $\mathcal{W}$, each D_j is contained in a member of $\mathcal{W}$. So (i) holds.

For each j, choose a continuous function $h_j : X \to [0, \infty)$ such that $h_j(x) > 0 \iff x \in \widetilde{O_j}$. Since the $\widetilde{O_j}$'s cover X,

$$H(x) := \sum_{j=1}^{J} h_j(x) > 0 \qquad (x \in X).$$

Define $\psi_j(x) := \frac{h_j(x)}{H(x)}$. Then $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_J) : X \to \Delta^{J-1}$ is an honest partition of unity for $(\widetilde{O_j})_{j=1}^J$, and $\operatorname{supp} \psi_j = \overline{\widetilde{O_j}} = D_j$. Thus, $\mathcal{H} = \big((\widetilde{O_j})_{j=1}^J, (D_j)_{j=1}^J, \boldsymbol{\psi}\big)$ is an honest partition structure.

Recall that $\operatorname{ord}_{\mathcal{D}}(x) = \sum_{j=1}^J \mathbf{1}_{D_j}(x) - 1$. Since $D_j \subset B_j$ for every j,

$$\operatorname{ord}_{\mathcal{D}}(x) \leq \sum_{j=1}^{J} \mathbf{1}_{B_j}(x) - 1.$$

Consequently,

$$\sup_{\mu \in \mathcal{P}_G(X)} \int_X \operatorname{ord}_{\mathcal{D}}(x)\, d\mu(x) \leq \sup_{\mu \in \mathcal{P}_G(X)} \int_X \operatorname{ord}_{\mathcal{B}}(x) d\mu(x) < a_0 < a.$$

This proves (iii).

For each j, choose $z_j \in B_j$ and denote $v_j := f_0(z_j) \in (0,1)^m$. We always have $z_j \in B_j \subset A_j$. If $\psi_j(x) > 0$, then $x \in \widetilde{O_j} \subset B_j \subset A_j$. Hence $\|v_j - f_0(x)\| = \|f_0(z_j) - f_0(x)\| < \varepsilon$. Since $\psi_j(x) \geq 0$ and $\sum_j \psi_j(x) = 1$,

$$\|f_*(x) - f_0(x)\| = \left\| \sum_{j=1}^{J} \psi_j(x)\big(v_j - f_0(x)\big) \right\| \leq \sum_{j=1}^{J} \psi_j(x) \|v_j - f_0(x)\| < \varepsilon.$$

Thus $\|f_* - f_0\|_\infty < \varepsilon$, and (ii) holds.

Finally, suppose that $\boldsymbol{\psi}(x) = \boldsymbol{\psi}(y)$. Choose j such that $\psi_j(x) > 0$. Then $\psi_j(y) > 0$ as well. Since the partition of unity is honest, $x, y \in \widetilde{O_j} \subset B_j \subset A_j$. As $\mathcal{A}$ refines $\mathcal{W}$, the points x and y lie in a common member of $\mathcal{W}$. $\square$

The following lemma converts a bound over invariant measures into a uniform bound along sufficiently invariant finite sets. We include a proof, using the empirical-measure argument related to [Mey26, Proposition 9.1].

Lemma 3.2. *Let $w : X \to \mathbb{R}$ be a bounded upper semicontinuous function and let $b \in \mathbb{R}$ satisfy $\sup_{\mu \in \mathcal{P}_G(X)} \int_X w\,d\mu < b$. Then there exist a finite set $K \subset G$ and $\delta > 0$ such that every nonempty finite (K, δ)-invariant set $T \subset G$ satisfies*

$$\sup_{x \in X} \frac{1}{|T|} \sum_{t \in T} w(tx) < b.$$

Proof. Suppose, towards a contradiction, that the conclusion fails. Choose an increasing sequence of finite sets $K_n \subset G$ with $e \in K_n$ and $\bigcup_n K_n = G$, and choose $\delta_n \downarrow 0$. For every n, there are a nonempty finite (K_n, δ_n)-invariant set $T_n \subset G$ and a point $x_n \in X$ such that

$$\frac{1}{|T_n|} \sum_{t \in T_n} w(tx_n) \geq b.$$

Define

$$\mu_n := \frac{1}{|T_n|} \sum_{t \in T_n} \delta_{tx_n} \in \mathrm{Prob}(X).$$

After passing to a subsequence, assume that $\mu_n \to \mu$ in the weak* topology. Then $\mu \in \mathcal{P}_G(X)$. Since w is bounded and upper semicontinuous, the Portmanteau theorem (see e.g. [Bil99, Theorem 2.1]) gives

$$b \leq \limsup_{n \to \infty} \int_X w\,d\mu_n \leq \int_X w\,d\mu \leq \sup_{\nu \in \mathcal{P}_G(X)} \int_X w\,d\nu,$$

contradicting the hypothesis. □

We now apply the preceding lemma to the function $\mathrm{ord}_{\mathcal{D}}$.

Corollary 3.3. *For the honest partition structure $\mathcal{H} = \big((\widetilde{O_j})_{j=1}^J, (D_j)_{j=1}^J, \boldsymbol{\psi}\big)$ obtained in Lemma 3.1, there exist a finite set $K_{\mathcal{H}} \subset G$, with $e \in K_{\mathcal{H}}$, and $\delta_{\mathcal{H}} \in (0,1)$ such that every nonempty finite $(K_{\mathcal{H}}, \delta_{\mathcal{H}})$-invariant set $T \subset G$ satisfies*

$$\sup_{x \in X} \sum_{t \in T} \mathrm{ord}_{\mathcal{D}}(tx) < a|T|.$$

Moreover, for every such T, let

$$\mathrm{Loc}_{\mathcal{D}}^T := \left\{ \big(\mathrm{loc}_{\mathcal{D}}(tx)\big)_{t \in T} : x \in X \right\},$$

and define

$$L_T := \bigcup_{(I_t)_{t \in T} \in \mathrm{Loc}_{\mathcal{D}}^T} \prod_{t \in T} \Delta_{I_t}.$$

Then

$$\boldsymbol{\psi}^T(X) \subset L_T, \qquad \dim L_T < a|T|.$$

Proof. The function $\operatorname{ord}_{\mathcal{D}}(x) = \sum_{j=1}^{J} \mathbf{1}_{D_j}(x) - 1$ is bounded and upper semicontinuous, since every D_j is closed. Lemma 3.1 gives

$$\sup_{\mu \in \mathcal{P}_G(X)} \int_X \operatorname{ord}_{\mathcal{D}}(x)\, d\mu(x) < a.$$

Applying Lemma 3.2 with $w = \operatorname{ord}_{\mathcal{D}}$ and $b = a$, we obtain a finite set $K_{\mathcal{H}} \subset G$ and $\delta_{\mathcal{H}} > 0$ such that for every nonempty finite $(K_{\mathcal{H}}, \delta_{\mathcal{H}})$-invariant set T,

$$\sup_{x \in X} \frac{1}{|T|} \sum_{t \in T} \operatorname{ord}_{\mathcal{D}}(tx) < a.$$

Enlarging $K_{\mathcal{H}}$ and decreasing $\delta_{\mathcal{H}}$, if necessary, we may assume that $e \in K_{\mathcal{H}}$, and $\delta_{\mathcal{H}} \in (0, 1)$. This proves the first assertion.

Fix such a set T. For $x \in X$ and $t \in T$, denote

$$I_t(x) := \operatorname{loc}_{\mathcal{D}}(tx) = \{j : tx \in D_j\}.$$

Since $\operatorname{supp} \psi_j \subset D_j$, we have $\boldsymbol{\psi}(tx) \in \Delta_{I_t(x)}$. Therefore

$$\boldsymbol{\psi}^T(x) \in \prod_{t \in T} \Delta_{I_t(x)},$$

and hence $\boldsymbol{\psi}^T(X) \subset L_T$. Since $\dim \Delta_{I_t(x)} = \operatorname{ord}_{\mathcal{D}}(tx)$, the dimension of the corresponding product face is $\sum_{t \in T} \operatorname{ord}_{\mathcal{D}}(tx)$.

For every pattern $(I_t)_{t \in T} \in \operatorname{Loc}_{\mathcal{D}}^T$, choose $x \in X$ which realizes it. Since $\prod_{t \in T} \Delta_{I_t}$ is a finite product of simplices, its dimension is the sum of the dimensions of its factors. Together with $\dim \Delta_{I_t} = \operatorname{ord}_{\mathcal{D}}(tx)$, we have

$$\dim \left(\prod_{t \in T} \Delta_{I_t} \right) = \sum_{t \in T} \dim \Delta_{I_t} = \sum_{t \in T} \operatorname{ord}_{\mathcal{D}}(tx) < a|T|.$$

Each product $\prod_{t \in T} \Delta_{I_t}$ is a face of the finite polytope $(\Delta^{J-1})^T$. Hence L_T is a finite union of faces of this polytope and is contained in the finite subcomplex generated by those faces. Consequently,

$$\dim L_T = \max_{(I_t) \in \operatorname{Loc}_{\mathcal{D}}^T} \dim \left(\prod_{t \in T} \Delta_{I_t} \right) < a|T|. \qquad \square$$

Thus the dynamical dimension hypothesis has been converted into a uniform dimension bound on sufficiently invariant finite orbit windows.

4. Recognition and generic embeddings

By the standard Baire category argument in order to prove Theorem A, it is enough to prove the following proposition.

Proposition 4.1. *Let G be a countably infinite discrete amenable group acting freely by homeomorphisms on a nonempty compact metrizable space X. Suppose that the action has URPC and that $\operatorname{mdim}(X, G) < m/2$. Then, for* *every* $f_{\mathrm{orig}} \in C(X,[0,1]^m)$ *and every* $\varepsilon, \eta > 0$*, there exists* $q \in C(X,[0,1]^m)$ *such that*

$$\|q - f_{\mathrm{orig}}\|_\infty < \varepsilon \qquad \textit{and} \qquad q^G \textit{ is an } \eta\textit{-embedding.}$$

Proof of Theorem A given Proposition 4.1. For $\eta > 0$, set

$$E_\eta := \left\{ f \in C(X,[0,1]^m) : f^G(x) \neq f^G(y) \text{ whenever } \rho(x,y) \geq \eta \right\}.$$

The set E_η is open in the uniform topology. Indeed, suppose that $f_n \notin E_\eta$ and $f_n \to f$ uniformly. Choose $x_n, y_n \in X$ such that

$$\rho(x_n, y_n) \geq \eta \qquad \text{and} \qquad f_n^G(x_n) = f_n^G(y_n).$$

After passing to a subsequence, we may assume that

$$x_n \to x \qquad \text{and} \qquad y_n \to y.$$

Then $\rho(x,y) \geq \eta$. For every $g \in G$,

$$f(gx) = \lim_{n\to\infty} f_n(gx_n) = \lim_{n\to\infty} f_n(gy_n) = f(gy),$$

so $f^G(x) = f^G(y)$. Hence $f \notin E_\eta$.

By Proposition 4.1, E_η is dense for every $\eta > 0$.

Finally,

$$\mathcal{E}_m(X,G) = \bigcap_{n=1}^{\infty} E_{1/n}.$$

Since $C(X,[0,1]^m)$ is complete in the uniform metric, the Baire category theorem implies that $\mathcal{E}_m(X,G)$ is a dense G_δ subset of $C(X,[0,1]^m)$. ☐

It remains to prove Proposition 4.1. Our aim is to perturb the given observable so that points with equal full orbit names[9] must be less than η apart. The construction has two main steps. First, we arrange that equality of full orbit names forces the two points into a common tower level or a common open set covering part of the remainder. To achieve this, we state a perturbation result which, for suitable compact–open pairs $K \subset O$, ensures that a point in K cannot have the same finite orbit name as a point outside O. We then use URPC to construct the pairs needed for the preceding conclusion.

Second, we perturb the observable further so that, within each of these common sets, equality of full orbit names forces the points to be less than η apart. For this step, we construct covers on which the relevant finite orbit maps have arbitrarily small oscillation, with order controlled by the mean dimension bound. These covers allow us to approximate the orbit maps on tower bases by maps that distinguish the required small sets, including the pieces of the remainder moved into the bases by comparison. Cutoff functions combine these approximations into a continuous global observable. The perturbation is small enough to preserve the conclusion of the first step,

[9]We call a pair of distinct points with equal full orbit names a **collision**.

and the two steps together give an η-embedding. The choices of parameters and the proof of the proposition are completed in Subsection 4.5.

4.1. **Recognition by a finite orbit name.** For a finite subset $F \subset G$, denote

$$R_F = F^{-1}F, \qquad \mathcal{Q}_F = R_F^{-1}R_F \setminus \{e\}.$$

Proposition 4.2. *Fix* $a < m/2$*, an honest partition structure* $\mathcal{H} = \big((\widetilde{O_j})_{j=1}^J, (D_j)_{j=1}^J, \boldsymbol{\psi}\big)$*, vectors* $v_1, \ldots, v_J \in (0,1)^m$*, and the affine observable*

$$f_*(x) := \sum_{j=1}^{J} v_j \psi_j(x).$$

Let $A \subset G$ *be finite, with* $e \in A$*, and assume that*

$$\sum_{c \in A} \operatorname{ord}_{\mathcal{D}}(cgx) < a|A| \qquad (g \in G,\ x \in X). \tag{4.1}$$

Then there exists a finite set $F \subset G$*, with* $e \in F$*, having the following property.*

Let $U \subset X$ *be open. Assume that the sets* tU*,* $t \in F$*, are pairwise disjoint and that*

$$U \cap qU = \emptyset \qquad q \in \mathcal{Q}_F. \tag{4.2}$$

Let $\Lambda \neq \emptyset$ *be finite, and suppose that, for every* $\ell \in \Lambda$*,* $K_\ell \subset X$ *is compact and* $O_\ell \subset X$ *is open, with* $K_\ell \subset O_\ell \Subset U$*, and that the closures* $\overline{O_\ell}$*,* $\ell \in \Lambda$*, are pairwise disjoint. Then, for every* $\varepsilon > 0$*, there exists* $q \in C(X, [0,1]^m)$ *such that* $\|q - f_*\|_\infty < \varepsilon$ *and*

$$q^F(K_\ell) \cap q^F(X \setminus O_\ell) = \emptyset \qquad (\ell \in \Lambda). \tag{4.3}$$

An observable supplied by Proposition 4.2 is called a **(finite-window) localizer**. Equation (4.3) is referred to as the **(finite-window) recognition property**. Proposition 4.2 is proved in Section 5, with the final construction given in Subsection 5.4.

Remark 4.3. The localizer q_0 in Proposition 4.2 is of the following form. There are continuous functions

$$\rho_\ell : X \to [0,1], \qquad \ell \in \Lambda,$$

with

$$\rho_\ell = 1 \text{ on } K_\ell, \qquad \operatorname{supp} \rho_\ell \subset O_\ell,$$

as well as a continuous map

$$Q : (\Delta^{J-1})^A \times [0,1]^{F^{-1} \times \Lambda} \longrightarrow \mathbb{R}^m$$

such that

$$q_0(z) = Q\big(\boldsymbol{\psi}^A(z), \boldsymbol{\rho}^{F^{-1}}(z)\big), \qquad z \in X,$$

where

$$\boldsymbol{\rho}(z) := (\rho_\ell(z))_{\ell \in \Lambda}.$$

The support properties of these functions and the above factorization will be used in Lemma 4.7.

4.2. **Placing collisions in a common tower level or a remainder neighbourhood.** We follow Naryshkin's type-semigroup notation [Nar24, Definition 3.7], a variant of Ma's generalized type semigroup [Ma21] in which no further quotient by mutual subequivalence is taken.

Let $V_1, \dots, V_r, W_1, \dots, W_s \subset X$ be open sets, and let $n_1, \dots, n_r, m_1, \dots, m_s \in \mathbb{N}_0$. In the formal sums

$$\sum_{i=1}^{r} n_i[V_i] \qquad \text{and} \qquad \sum_{j=1}^{s} m_j[W_j],$$

the coefficients specify the numbers of labelled copies of the corresponding open sets. We call (i,a), where $1 \le i \le r$ and $1 \le a \le n_i$, a **source label**, and (j,b), where $1 \le j \le s$ and $1 \le b \le m_j$, a **target label**.

We write

$$\sum_{i=1}^{r} n_i[V_i] \prec \sum_{j=1}^{s} m_j[W_j]$$

if, for every choice of compact sets $K_{i,a} \subset V_i$, one for each source label (i,a), there exist finite families $\mathcal{U}_{i,a}$ of open subsets of X satisfying

$$K_{i,a} \subset \bigcup_{U \in \mathcal{U}_{i,a}} U,$$

together with, for each $U \in \mathcal{U}_{i,a}$, an element $g_{i,a,U} \in G$ and a target label $\tau(i,a,U)$, such that:

(i) if $\tau(i,a,U) = (j,b)$, then $g_{i,a,U}U \subset W_j$;
(ii) for each fixed target label (j,b), the translated sets $g_{i,a,U}U$, as (i,a,U) ranges over all triples satisfying $\tau(i,a,U) = (j,b)$, are pairwise disjoint.

We shall use without further comment that this relation is additive and transitive, and that

$$n_i \le m_i \text{ for all } i \quad \Longrightarrow \quad \sum_i n_i[V_i] \prec \sum_i m_i[V_i].$$

Lemma 4.4. [Nar24, Lemma 3.19] *Let G be a countably infinite discrete amenable group acting freely by homeomorphisms on a nonempty compact metrizable space X. Let $U \subset X$ be a sweep-out set. Then there exist $\theta_U > 0$ and a finite set $K_U \subset G$ such that if $\{(S_i, V_i)\}_{i=1}^{r}$ is an open castle and every shape S_i is (K_U, θ_U)-invariant, then*

$$\sum_{i=1}^{r} \lceil \theta_U |S_i| \rceil \, [V_i] \prec [U].$$

The next proposition is extracted from and adapted to the present finite-window setting from the mechanism in Naryshkin's proof of the shift-embedding theorem [Nar24, Subsection 6.1].

Proposition 4.5. *Let $U \subset X$ be a sweep-out set, and let $\theta_U > 0$ and the finite set $K_U \subset G$ be supplied by Lemma 4.4. Fix finite sets $F, K \subset G$, a number $\alpha > 0$, and a (K, α)-URPC witness*

$$\mathfrak{W}^0 = \left(\{S_i\}_{i=1}^r, \{V_i^0\}_{i=1}^r, \{g_j\}_{j=1}^N, \{U_j^0\}_{j=1}^N\right).$$

Here S_i and V_i^0 are the tower shapes and bases, g_j are the transporters, and U_j^0 are the open sets covering the remainder. Let

$$J_{i,c}, \qquad 1 \le i \le r, \quad 1 \le c \le \lfloor \alpha |S_i| \rfloor,$$

be the partition of $\{1, \dots, N\}$ associated with this witness. For each j, write $i(j)$ and $c(j)$ for the unique indices such that $j \in J_{i(j),c(j)}$.

Assume that every S_i is also (K_U, θ_U)-invariant and that

$$1 + \lfloor \alpha |S_i| \rfloor \le \lceil \theta_U |S_i| \rceil \qquad (1 \le i \le r). \tag{4.4}$$

Then there exist open sets [10]

$$V_i^- \Subset V_i^+ \Subset B_i \Subset V_i^0 \qquad (1 \le i \le r),$$

and

$$U_j^- \Subset U_j^+ \Subset U_j^0 \qquad (1 \le j \le N),$$

such that

$$\mathfrak{W}^\pm := \left(\{S_i\}_{i=1}^r, \{V_i^\pm\}_{i=1}^r, \{g_j\}_{j=1}^N, \{U_j^\pm\}_{j=1}^N\right)$$

are both (K, α)-URPC witnesses with the same shapes, transporters, and index partition as $\mathfrak{W}^0$. Moreover:

(i) *For every $1 \le j \le N$,*

$$g_j \overline{U_j^+} \subset V_{i(j)}^+.$$

(ii) *There exist a nonempty finite index set Λ, compact sets $K_\ell \subset X$, and open sets $O_\ell \subset X$, such that*

$$K_\ell \subset O_\ell \Subset U \qquad (\ell \in \Lambda),$$

and the closures $\overline{O_\ell}$ are pairwise disjoint. If $q \in C(X, [0,1]^m)$ satisfies the finite-window recognition property

$$q^F(K_\ell) \cap q^F(X \setminus O_\ell) = \emptyset \qquad (\ell \in \Lambda), \tag{4.5}$$

then any two points $x, y \in X$ with $q^G(x) = q^G(y)$ belong to a common member of the finite open cover

$$\mathcal{C}^+ := \{sV_i^+ : 1 \le i \le r,\ s \in S_i\} \cup \{U_j^+ : 1 \le j \le N\}.$$

Each pair (K_ℓ, O_ℓ) in Proposition 4.5 is called a **routing pair**, and O_ℓ is its open **buffer**. We call the witnesses $\mathfrak{W}^-$ and $\mathfrak{W}^+$, the intermediate bases B_i, and the routing pairs the **buffered URPC data** associated with $\mathfrak{W}^0$.

[10] The sets B_i will be used to construct the cutoff functions in Subsection 4.4.

Proof. We first construct the nested witnesses and the intermediate bases. Since $\mathfrak{W}^0$ is a URPC witness, its tower levels and remainder sets form a finite open cover of X. Choose compact sets

$$E_{i,s} \subset sV_i^0, \qquad H_j \subset U_j^0,$$

whose union is X. For each j, choose an open set U_j^+ such that

$$H_j \subset U_j^+ \Subset U_j^0.$$

For each i, the compact set

$$\bigcup_{s\in S_i} s^{-1}E_{i,s} \;\cup \bigcup_{\{j:i(j)=i\}} g_j\overline{U_j^+}$$

is contained in V_i^0. Choose an open neighbourhood V_i^+ of this compact set with $V_i^+ \Subset V_i^0$.

The sets sV_i^+ and U_j^+ still cover X, since they contain $E_{i,s}$ and H_j, respectively. The castle disjointness is inherited from $\mathfrak{W}^0$, and

$$g_j\overline{U_j^+} \subset V_{i(j)}^+.$$

For each fixed (i,c), the transported closures $g_j\overline{U_j^+}$, $j \in J_{i,c}$, are pairwise disjoint, since they lie in the pairwise disjoint sets $g_jU_j^0$. Thus $\mathfrak{W}^+$ is a (K,α)-URPC witness with the original shapes, transporters, and index partition.

Apply the same construction to $\mathfrak{W}^+$ to obtain $\mathfrak{W}^-$, with

$$V_i^- \Subset V_i^+, \qquad U_j^- \Subset U_j^+.$$

Finally, choose open sets B_i satisfying

$$V_i^+ \Subset B_i \Subset V_i^0.$$

We next use comparison to move pieces of all the bases V_i^+ and remainder sets U_j^+ into U, with disjoint images. For each fixed (i,c), the sets $g_jU_j^+$, $j \in J_{i,c}$, are pairwise disjoint subsets of V_i^+. Therefore

$$\sum_{j=1}^{N}[U_j^+] \prec \sum_{i=1}^{r}\lfloor\alpha|S_i|\rfloor[V_i^+].$$

By additivity, (4.4), and Lemma 4.4,

$$\begin{aligned}\sum_{i=1}^{r}[V_i^+] + \sum_{j=1}^{N}[U_j^+] &\prec \sum_{i=1}^{r}\bigl(1+\lfloor\alpha|S_i|\rfloor\bigr)[V_i^+]\\ &\prec \sum_{i=1}^{r}\lceil\theta_U|S_i|\rceil\,[V_i^+] \prec [U].\end{aligned}$$

The last comparison applies because $\{(S_i, V_i^+)\}_{i=1}^r$ is an open castle with (K_U, θ_U)-invariant shapes. In particular,

$$\sum_{i=1}^{r}[V_i^+] + \sum_{j=1}^{N}[U_j^+] \prec [U]. \tag{4.6}$$

Since $\mathfrak{W}^-$ is a URPC witness, choose compact sets

$$C_{i,s} \subset sV_i^-, \qquad D_j \subset U_j^-,$$

such that

$$X = \bigcup_{i=1}^{r} \bigcup_{s\in S_i} C_{i,s} \cup \bigcup_{j=1}^{N} D_j. \tag{4.7}$$

For each i, put

$$C_i := \bigcup_{s\in S_i} s^{-1}C_{i,s}.$$

Then C_i is compact and contained in V_i^-.

Apply (4.6) to the compact source sets $C_i \subset V_i^+$ and $D_j \subset U_j^+$. Intersect the implementing open sets with their respective source sets V_i^+ or U_j^+. By compactness, shrink these open sets while retaining covers of C_i and D_j, and then choose compact shrinkings of those covers. This gives integers $L_i, M_j \geq 0$, compact sets $A^V_{i,a}, A^U_{j,b}$, open sets $P^V_{i,a}, P^U_{j,b}$, and elements $r^V_{i,a}, r^U_{j,b} \in G$, satisfying

$$C_i \subset \bigcup_{a=1}^{L_i} A^V_{i,a}, \qquad A^V_{i,a} \subset P^V_{i,a} \Subset V_i^+, \qquad 1 \leq i \leq r, \quad 1 \leq a \leq L_i,$$
$$D_j \subset \bigcup_{b=1}^{M_j} A^U_{j,b}, \qquad A^U_{j,b} \subset P^U_{j,b} \Subset U_j^+, \qquad 1 \leq j \leq N, \quad 1 \leq b \leq M_j,$$

such that all the sets

$$r^V_{i,a}\overline{P^V_{i,a}} \quad \text{and} \quad r^U_{j,b}\overline{P^U_{j,b}}$$

form one pairwise disjoint family of compact subsets of U.

Define

$$\Lambda := \{(V, i, a) : 1 \leq i \leq r,\ 1 \leq a \leq L_i\} \\ \sqcup \{(U, j, b) : 1 \leq j \leq N,\ 1 \leq b \leq M_j\}.$$

For $\ell = (V, i, a)$, set

$$K_\ell := r^V_{i,a}A^V_{i,a}, \qquad O_\ell := r^V_{i,a}P^V_{i,a},$$

and for $\ell = (U, j, b)$, set

$$K_\ell := r^U_{j,b}A^U_{j,b}, \qquad O_\ell := r^U_{j,b}P^U_{j,b}.$$

These pairs satisfy $K_\ell \subset O_\ell \Subset U$, and the closures $\overline{O_\ell}$ are pairwise disjoint. Since the sets $C_{i,s}$ and D_j cover the nonempty space X, at least one C_i or D_j is nonempty. Hence at least one L_i or M_j is positive, so $\Lambda \neq \emptyset$.

It remains to verify the conclusion concerning equal orbit names. Suppose that q satisfies the finite-window recognition property (4.5) and that $q^G(x) = q^G(y)$. Then

$$q^G(hx) = q^G(hy) \qquad (h \in G),$$

because $q(ghx) = q(ghy)$ for every $g, h \in G$.

If $x \in C_{i,s}$, put $u = s^{-1}x$ and $v = s^{-1}y$. Choose a such that $u \in A^V_{i,a}$, and let $\ell = (V, i, a)$. Then

$$r^V_{i,a}u \in K_\ell, \qquad q^F(r^V_{i,a}u) = q^F(r^V_{i,a}v).$$

The finite-window recognition property (4.5) implies $r^V_{i,a}v \in O_\ell$, and hence $v \in P^V_{i,a}$. Since

$$u \in A^V_{i,a} \subset P^V_{i,a} \subset V_i^+,$$

we obtain $u, v \in V_i^+$, and therefore $x, y \in sV_i^+$.

If $x \in D_j$, choose b such that $x \in A^U_{j,b}$, and let $\ell = (U, j, b)$. Then

$$r^U_{j,b}x \in K_\ell, \qquad q^F(r^U_{j,b}x) = q^F(r^U_{j,b}y).$$

The finite-window recognition property (4.5) gives $r^U_{j,b}y \in O_\ell$, so $y \in P^U_{j,b}$. Together with

$$x \in A^U_{j,b} \subset P^U_{j,b} \subset U_j^+,$$

this yields $x, y \in U_j^+$.

The compact sets $C_{i,s}$ and D_j cover X by (4.7), so these cases exhaust all possibilities. Thus x and y belong to a common member of $\mathcal{C}^+$. □

4.3. **Covers of controlled order and small oscillation.** Fix an observable q_0 with the representation recorded in Remark 4.3. For a sufficiently invariant finite set S, we will construct covers on which q_0^S has arbitrarily small oscillation, while their order stays bounded linearly in $|S|$.

Lemma 4.6. *Let $F, R \subset G$ be finite, with $R \neq \emptyset$, and let $U \subset X$ be open. Suppose that the sets rU, $r \in R$, are pairwise disjoint. Let $\kappa, \varepsilon_R > 0$ satisfy $(1 + \varepsilon_R)/|R| < \kappa$. If a nonempty finite set $S \subset G$ satisfies $|RF^{-1}S| < (1 + \varepsilon_R)|S|$, then, for every $x \in X$,*

$$\big|\{u \in F^{-1}S : ux \in U\}\big| < \kappa|S|. \tag{4.8}$$

Proof. Fix $x \in X$ and put

$$T_x := \{u \in F^{-1}S : ux \in U\}.$$

We claim that the multiplication map

$$R \times T_x \longrightarrow RF^{-1}S, \qquad (r, u) \longmapsto ru,$$

is injective. Indeed, suppose that $r_1u_1 = r_2u_2$ for $r_1, r_2 \in R$ and $u_1, u_2 \in T_x$. Since $u_1x, u_2x \in U$, we have $r_1u_1x = r_2u_2x \in r_1U \cap r_2U$. The translates rU, $r \in R$, are pairwise disjoint, so $r_1 = r_2$. As $r_1u_1 = r_2u_2$, we conclude $u_1 = u_2$.

Hence $|R||T_x| \le |RF^{-1}S|$. Using the two hypotheses above, we obtain $|T_x| < \frac{1+\varepsilon_R}{|R|}|S| < \kappa|S|$. This proves (4.8). $\square$

Fix a with $\operatorname{mdim}(X,G) < a < m/2$ and a finite open cover $\mathcal{W}$ of X. Let

$$\mathcal{H} = \big((\widetilde{O}_j)_{j=1}^J, (D_j)_{j=1}^J, \boldsymbol{\psi}\big), \qquad v_1, \dots, v_J \in (0,1)^m,$$

be supplied by Lemma 3.1, and put

$$f_*(z) := \sum_{j=1}^J v_j \psi_j(z).$$

Let $K_{\mathcal{H}} \subset G$ and $\delta_{\mathcal{H}} \in (0,1)$ be the corresponding finite-invariance data from Corollary 3.3.

Fix a finite set $A \subset G$ containing e and satisfying (4.1), and let $F \subset G$ be supplied by Proposition 4.2. Let $U \subset X$ be an open sweep-out set, and let $(K_\ell, O_\ell)_{\ell \in \Lambda}$ be a nonempty finite family satisfying the hypotheses of that proposition. Fix a corresponding localizer q_0, together with the continuous functions $\rho_\ell : X \to [0,1]$ from Remark 4.3. Recall that

$$\rho_\ell = 1 \text{ on } K_\ell, \qquad \operatorname{supp} \rho_\ell \subset O_\ell \qquad (\ell \in \Lambda).$$

Write

$$\boldsymbol{\rho} : X \longrightarrow [0,1]^\Lambda, \qquad \boldsymbol{\rho}(z) := (\rho_\ell(z))_{\ell \in \Lambda},$$

and, for each finite set $E \subset G$, set

$$\boldsymbol{\rho}^E(z) := \big(\rho_\ell(uz)\big)_{(u,\ell) \in E \times \Lambda}.$$

Lemma 4.7. *With the data fixed above, let $\varepsilon_A, \kappa, \varepsilon_R > 0$, and let $R \subset G$ be a nonempty finite set such that the translates rU, $r \in R$, are pairwise disjoint and*

$$\frac{1+\varepsilon_R}{|R|} < \kappa.$$

Let $S \subset G$ be a nonempty finite set such that AS is $(K_{\mathcal{H}}, \delta_{\mathcal{H}})$-invariant and

$$|AS| < (1+\varepsilon_A)|S|, \qquad |RF^{-1}S| < (1+\varepsilon_R)|S|.$$

Define

$$\Pi_S : X \longrightarrow (\Delta^{J-1})^{AS} \times [0,1]^{F^{-1}S \times \Lambda},$$
$$\Pi_S(z) := \big(\boldsymbol{\psi}^{AS}(z), \boldsymbol{\rho}^{F^{-1}S}(z)\big),$$

and put $P_S := \Pi_S(X)$. Then there exists a continuous map

$$\overline{q}_S : P_S \longrightarrow ([0,1]^m)^S$$

such that

$$q_0^S = \overline{q}_S \circ \Pi_S, \tag{4.9}$$

and

$$\dim P_S < \big(a(1+\varepsilon_A) + \kappa\big)|S|. \tag{4.10}$$

Moreover, for every compact set $Y \subset X$ and every $\tau > 0$, there exists a finite relatively open cover $\mathcal{U}$ of Y such that:

(i) *$\mathcal{U}$ refines*

$$\bigvee_{s \in S} s^{-1}\mathcal{W}\bigg|_Y;$$

(ii) *the oscillation of q_0^S on each member of $\mathcal{U}$ is less than τ;*

(iii)

$$\operatorname{ord}(\mathcal{U}) < \big(a(1+\varepsilon_A)+\kappa\big)|S|.$$

In particular, P_S, Π_S, $\overline{q}_S$, and the stated dimension bound are fixed before the oscillation scale τ is chosen.

Proof. By Remark 4.3, there exists a continuous map

$$Q : (\Delta^{J-1})^A \times [0,1]^{F^{-1}\times\Lambda} \longrightarrow \mathbb{R}^m$$

such that

$$q_0(z) = Q\big(\boldsymbol{\psi}^A(z), \boldsymbol{\rho}^{F^{-1}}(z)\big), \qquad z \in X. \tag{4.11}$$

For $s \in S$, applying (4.11) to sz gives

$$q_0(sz) = Q\left(\big(\boldsymbol{\psi}(csz)\big)_{c\in A}, \big(\rho_\ell(usz)\big)_{(u,\ell)\in F^{-1}\times\Lambda}\right).$$

Since $cs \in AS$ and $us \in F^{-1}S$, all the entries on the right-hand side are coordinates of $\Pi_S(z)$. More explicitly, write a point of $(\Delta^{J-1})^{AS}\times[0,1]^{F^{-1}S\times\Lambda}$ as (ξ,η), where $\xi = (\xi_v)_{v\in AS}$, and $\eta = (\eta_{v,\ell})_{(v,\ell)\in F^{-1}S\times\Lambda}$. Define

$$\widetilde{q}_S(\xi,\eta) := \Big(Q\left((\xi_{cs})_{c\in A}, (\eta_{us,\ell})_{(u,\ell)\in F^{-1}\times\Lambda}\right)\Big)_{s\in S}.$$

Then $\widetilde{q}_S$ is continuous and $q_0^S = \widetilde{q}_S \circ \Pi_S$. Restricting $\widetilde{q}_S$ to $P_S = \Pi_S(X)$, we obtain a continuous map $\overline{q}_S := \widetilde{q}_S|_{P_S} : P_S \to ([0,1]^m)^S$ satisfying $q_0^S = \overline{q}_S \circ \Pi_S$. This proves (4.9).

We next estimate the dimension of P_S to prove (4.10). For $z \in X$, set $E_z := \{u \in F^{-1}S : uz \in U\}$. By Lemma 4.6,

$$|E_z| < \kappa|S|. \tag{4.12}$$

For $z \in X$, define the set of active localizing roots by[11]

$$E_z^{\mathrm{loc}} := \big\{u \in F^{-1}S : \rho_\ell(uz) \neq 0 \text{ for some } \ell \in \Lambda\big\} \subset E_z.$$

Therefore, by (4.12),

$$|E_z^{\mathrm{loc}}| < \kappa|S|. \tag{4.13}$$

Moreover, for every $u \in E_z^{\mathrm{loc}}$, there is a unique index $\lambda_z(u) \in \Lambda$ such that $\rho_{\lambda_z(u)}(uz) \neq 0$. Indeed, if both $\rho_\ell(uz)$ and $\rho_{\ell'}(uz)$ were nonzero, then $uz \in O_\ell \cap O_{\ell'}$, and the pairwise disjointness of the sets O_ℓ would imply $\ell = \ell'$. Consequently, $\boldsymbol{\rho}^{F^{-1}S}(z)$ has fewer than $\kappa|S|$ nonzero scalar coordinates. For each fixed set of positions in which nonzero coordinates may occur, these

[11] Since $\operatorname{supp}\rho_\ell \subset O_\ell \subset U$ for every $\ell \in \Lambda$, it holds $E_z^{\mathrm{loc}} \subset E_z$.

vectors lie in a coordinate cube whose dimension is the number of those positions. We now define these cubes.

For a subset $E \subset F^{-1}S$ and a map $\lambda : E \to \Lambda$, let

$$\Gamma(E,\lambda) := \{(u,\lambda(u)) : u \in E\} \subset F^{-1}S \times \Lambda.$$

Define

$$Z(E,\lambda) := \left\{ w \in [0,1]^{F^{-1}S\times\Lambda} : w_{u,\ell} = 0 \text{ whenever } (u,\ell) \notin \Gamma(E,\lambda) \right\}.$$

Thus only the coordinates indexed by $\Gamma(E,\lambda)$ are allowed to vary, and the coordinate projection

$$Z(E,\lambda) \longrightarrow [0,1]^E, \qquad w \longmapsto \big(w_{u,\lambda(u)}\big)_{u\in E},$$

is a homeomorphism. In particular,

$$\dim Z(E,\lambda) = |E|.$$

We now collect all support patterns allowed by (4.13). Set

$$Z_S := \bigcup_{\substack{E\subset F^{-1}S,\ |E|<\kappa|S| \\ \lambda:E\to\Lambda}} Z(E,\lambda).$$

This is a finite union, since both $F^{-1}S$ and Λ are finite. For every $z \in X$, the active coordinates of $\boldsymbol{\rho}^{F^{-1}S}(z)$ are precisely those indexed by $\Gamma(E_z^{\mathrm{loc}},\lambda_z)$. Hence

$$\boldsymbol{\rho}^{F^{-1}S}(z) \in Z(E_z^{\mathrm{loc}},\lambda_z) \subset Z_S,$$

and therefore

$$\boldsymbol{\rho}^{F^{-1}S}(X) \subset Z_S.$$

It remains to estimate the dimension of Z_S. Each cube $Z(E,\lambda)$ is compact, hence closed in Z_S, and has dimension $|E| < \kappa|S|$. Since Z_S is a finite union of these cubes, the finite sum theorem for covering dimension of closed sets gives

$$\dim Z_S = \max_{\substack{E\subset F^{-1}S,\ |E|<\kappa|S| \\ \lambda:E\to\Lambda}} \dim Z(E,\lambda) < \kappa|S|.$$

By hypothesis, the finite set AS is $(K_{\mathcal{H}},\delta_{\mathcal{H}})$-invariant. Hence Corollary 3.3, applied with $T = AS$, gives a finite union L_{AS} of products of simplex faces such that

$$\boldsymbol{\psi}^{AS}(X) \subset L_{AS} \qquad \text{and} \qquad \dim L_{AS} < a|AS|.$$

Using $|AS| < (1+\varepsilon_A)|S|$, we therefore obtain

$$\dim L_{AS} < a|AS| < a(1+\varepsilon_A)|S|.$$

Since $\Pi_S(z) = \big(\boldsymbol{\psi}^{AS}(z), \boldsymbol{\rho}^{F^{-1}S}(z)\big)$, the two inclusions above imply, for every $z \in X$, $\Pi_S(z) \in L_{AS} \times Z_S$. Consequently,

$$P_S = \Pi_S(X) \subset L_{AS} \times Z_S.$$

Since $P_S \subset L_{AS} \times Z_S$, monotonicity and the product theorem for covering dimension of metrizable spaces give

$$\dim P_S \le \dim(L_{AS} \times Z_S) \le \dim L_{AS} + \dim Z_S < \big(a(1+\varepsilon_A) + \kappa\big)|S|.$$

This proves (4.10).

Fix a compact set $Y \subset X$, and set $P_Y := \Pi_S(Y) \subset P_S$. We first show that every fibre of $\pi := \Pi_S|_Y : Y \to P_Y$ is contained in a single member of the restricted S-join $\bigvee_{s\in S} s^{-1}\mathcal{W}\big|_Y$. Indeed, fix $p \in P_Y$. Since $e \in A$, for every $s \in S$ we have $s = es \in AS$. Hence the coordinate $\boldsymbol{\psi}(sz)$ is recorded by $\boldsymbol{\psi}^{AS}(z)$, and therefore it is constant as z ranges over the fibre $\pi^{-1}(p) = \Pi_S^{-1}(p) \cap Y$. Denote this common vector by $\xi_s \in \Delta^{J-1}$. For each $s \in S$, choose $j_s \in \{1, \dots, J\}$ such that $(\xi_s)_{j_s} > 0$. By honesty of the partition of unity, for any $z \in \pi^{-1}(p)$,

$$\psi_{j_s}(sz) = (\xi_s)_{j_s} > 0 \quad \Longrightarrow \quad sz \in \widetilde{O_{j_s}}.$$

Thus,

$$\pi^{-1}(p) \subset U_p := Y \cap \bigcap_{s\in S} s^{-1}\widetilde{O_{j_s}}.$$

For each $s \in S$, choose $W_s \in \mathcal{W}$ such that $\widetilde{O_{j_s}} \subset D_{j_s} \subset W_s$. Then $U_p \subset Y \cap \bigcap_{s\in S} s^{-1}W_s$, and the set on the right is a member of $\bigvee_{s\in S} s^{-1}\mathcal{W}\big|_Y$. Thus each fibre of $\Pi_S|_Y$ is contained in one member of the restricted S-join.

For every $p \in P_Y$, choose a member U_p of the restricted S-join which contains $\pi^{-1}(p)$. Since Y is compact and P_Y is Hausdorff, π is a closed map. Therefore $V_p := P_Y \setminus \pi(Y \setminus U_p)$ is an open neighbourhood of p satisfying $\pi^{-1}(V_p) \subset U_p$. By continuity of $\overline{q}_S|_{P_Y}$, choose an open neighbourhood $W_p \subset P_Y$ of p on which $\overline{q}_S$ has oscillation less than τ. A finite subfamily of $\{V_p \cap W_p : p \in P_Y\}$ covers P_Y.

Since

$$\dim P_Y \le \dim P_S < \big(a(1+\varepsilon_A) + \kappa\big)|S|,$$

the definition of covering dimension yields a finite open refinement $\mathcal{R}$ of this cover such that

$$\operatorname{ord}(\mathcal{R}) \le \dim P_Y < \big(a(1+\varepsilon_A) + \kappa\big)|S|.$$

Set $\mathcal{U} := \{\pi^{-1}(R) : R \in \mathcal{R}\}$, viewed as a relatively open cover of Y. This cover refines the restricted S-join, and, since $q_0^S = \overline{q}_S \circ \pi$, the oscillation of q_0^S on each member is less than τ. Finally,

$$\operatorname{ord}(\mathcal{U}) \le \operatorname{ord}(\mathcal{R}) < \big(a(1+\varepsilon_A) + \kappa\big)|S|.$$

This proves all assertions. $\square$

4.4. **The global perturbation.** The preceding constructions play two complementary roles. The finite-window recognition property of q_0, together with the routing pairs, ensures that any collision of a sufficiently small perturbation of q_0 is routed into a common tower or remainder chart. On the other hand, Lemma 4.7 produces, on every sufficiently invariant shape S, a low-order cover on which the block q_0^S has arbitrarily small oscillation.

On each buffered tower base we perturb $q_0^{S_i}$ to a nearby map encoding the low-order cover supplied by Lemma 4.7 and the transported remainder pieces. Cutoff functions supported inside the buffered bases combine these block maps with q_0 into a single global observable q. The support conditions ensure continuity, and the small perturbation preserves finite-window recognition. Since the castle levels are pairwise disjoint, at most one tower cutoff contributes at any point.

The local-to-global tower assembly here parallels the corresponding construction in Gutman–Tsukamoto [GT14, Lemma 2.1 and the proof of Proposition 3.1]. There, block maps defined on clopen tower bases are placed along the corresponding tower levels to obtain a global observable. In our setting the tower levels are only open and a remainder may be present, so this direct piecewise assembly is unavailable. Buffered cutoffs provide continuity, while finite-window recognition and comparison handle the routing and the remainder.

Lemma 4.8. *Let Y be a compact metrizable space, let $F : Y \to [0,1]^N$ be continuous, and let $\delta > 0$. Let $\mathcal{U}$ be a finite open cover of Y and $\mathcal{V}$ a finite family of open subsets of Y, with repetitions discarded. Assume that*[12]

$$\mathrm{mult}(\mathcal{U}) + \mathrm{mult}(\mathcal{V}) < \frac{N}{2},$$

and that the oscillation of F on each member of $\mathcal{U}$ is less than $\delta/2$. Then there exists a continuous map $\widetilde{F} : Y \to [0,1]^N$ such that

$$\|\widetilde{F} - F\|_\infty < \delta$$

and $\widetilde{F}$ encodes every member of $\mathcal{U} \cup \mathcal{V}$.

Proof. Choose $\gamma \in (0,1)$ sufficiently small that

$$\mathrm{mult}(\mathcal{U}) + \mathrm{mult}(\mathcal{V}) \le (1-\gamma)N/2.$$

Apply [Nar24, Lemma 6.14] to $\mathcal{U}$ and $\mathcal{V}$. Taking the full coordinate set $I = \{1, \dots, N\}$, for which $|I| = N > (1-\gamma)N$, gives the required approximation and encoding properties. ∎

Proposition 4.9. *Fix $\eta > 0$, and let $\mathcal{W}$ be a finite open cover of X with* $\mathrm{mesh}(\mathcal{W}) < \eta$. *Retain the data associated with the prepared localizer q_0 from Lemma 4.7, in particular $F, U, R, \varepsilon_A, \kappa$, and ε_R.*

[12] Naryshkin's notion of order in [Nar24, Definition 6.8] is our multiplicity.

Let $\alpha > 0$, and suppose that we are given buffered URPC data as in Proposition 4.5, with shapes S_i, tower bases $V_i^- \Subset V_i^+ \Subset B_i$, remainder sets $U_j^- \Subset U_j^+$, transporters g_j, and routing pairs

$$K_\ell \subset O_\ell \Subset U, \qquad \ell \in \Lambda.$$

Denote

$$N_{\mathcal{W}} := \mathrm{mult}(\mathcal{W}) = \mathrm{ord}(\mathcal{W}) + 1 \qquad \textit{and} \qquad d_0 := a(1+\varepsilon_A) + \kappa + N_{\mathcal{W}}\alpha.$$

Suppose that the following conditions hold:

(i) $q_0^F(K_\ell) \cap q_0^F(X \setminus O_\ell) = \emptyset$, *$\ell \in \Lambda$.*
(ii) $d_0 < d < m/2$ *for some* $d \in \mathbb{R}$.
(iii) *Every shape S_i satisfies $|AS_i| < (1+\varepsilon_A)|S_i|$, the set AS_i is $(K_{\mathcal{H}}, \delta_{\mathcal{H}})$-invariant, $|RF^{-1}S_i| < (1+\varepsilon_R)|S_i|$, and $|S_i| > \frac{1}{d-d_0}$.*

Then, for every $\varepsilon > 0$, there exists $q \in C(X,[0,1]^m)$ such that $\|q - q_0\|_\infty < \varepsilon$ and q^G is an η-embedding.

Proof. For every $\ell \in \Lambda$, the compact sets $q_0^F(K_\ell)$ and $q_0^F(X \setminus O_\ell)$ are disjoint. Since Λ is finite, we may set

$$\sigma := \min_{\ell \in \Lambda} \mathrm{dist}\big(q_0^F(K_\ell), q_0^F(X \setminus O_\ell)\big) > 0.$$

Choose $0 < \delta < \min\{\varepsilon, \sigma/3\}$.

For each i, denote $Y_i := \overline{B_i}$. Applying Lemma 4.7 with $S = S_i$, $Y = Y_i$, and $\tau = \delta/2$, we obtain a finite relatively open cover $\mathcal{U}_i$ of Y_i which refines $\bigvee_{s \in S_i} s^{-1}\mathcal{W}\big|_{Y_i}$, such that $q_0^{S_i}$ has oscillation less than $\delta/2$ on every member of $\mathcal{U}_i$, and

$$\mathrm{ord}(\mathcal{U}_i) < \big(a(1+\varepsilon_A) + \kappa\big)|S_i|. \tag{4.14}$$

For each base index i, use all remainder indices j with $i(j) = i$ and all members $W \in \mathcal{W}$ to form the finite family

$$\mathcal{V}_i := \Big\{ Y_i \cap g_j(U_j^+ \cap W) : 1 \le j \le N,\ i(j) = i,\ W \in \mathcal{W} \Big\} \setminus \{\emptyset\}.$$

Repeated subsets occur only once. At a given point $y \in Y_i$, each color contributes at most one j, because the sets $g_j U_j^+$ are disjoint within that color. For this j, $g_j^{-1} y$ belongs to at most $N_{\mathcal{W}} = \mathrm{mult}(\mathcal{W})$ members of $\mathcal{W}$. Thus

$$\mathrm{mult}(\mathcal{V}_i) \le N_{\mathcal{W}} \lfloor \alpha |S_i| \rfloor \le N_{\mathcal{W}} \alpha |S_i|.$$

By (4.14) and $\mathrm{mult}(\mathcal{U}_i) = \mathrm{ord}(\mathcal{U}_i) + 1$,

$$\begin{aligned} \mathrm{mult}(\mathcal{U}_i) + \mathrm{mult}(\mathcal{V}_i) &< \big(a(1+\varepsilon_A) + \kappa + N_{\mathcal{W}}\alpha\big)|S_i| + 1 \\ &= d_0 |S_i| + 1. \end{aligned}$$

Assumption (iii) gives $(d - d_0)|S_i| > 1$, so, using (ii),

$$\mathrm{mult}(\mathcal{U}_i) + \mathrm{mult}(\mathcal{V}_i) < d_0|S_i| + 1 < d|S_i| < \frac{m|S_i|}{2}.$$

This is the hypothesis of Lemma 4.8 with $N = m|S_i|$.

Define

$$Q_i : Y_i \longrightarrow ([0,1]^m)^{S_i}, \qquad Q_i(u) := (q_0(su))_{s\in S_i}.$$

Apply Lemma 4.8 with $N = m|S_i|$, $\mathcal{U} = \mathcal{U}_i$, and $\mathcal{V} = \mathcal{V}_i$. We obtain a continuous map $\widetilde{Q}_i = (\widetilde{Q}_i^s)_{s\in S_i} : Y_i \to ([0,1]^m)^{S_i}$ such that

$$\|\widetilde{Q}_i - Q_i\|_\infty < \delta$$

and $\widetilde{Q}_i$ encodes every member of $\mathcal{U}_i \cup \mathcal{V}_i$.

In particular, encoding $\mathcal{U}_i$ and its refinement of the S_i-join give

$$\widetilde{Q}_i(u) = \widetilde{Q}_i(v) \quad\Longrightarrow\quad \rho(su, sv) < \eta \quad (s \in S_i).$$

Thus each block map is a local η-embedding for the orbit metric $d_{S_i}(u,v) := \max_{s\in S_i} \rho(su, sv)$.

Choose $\chi_i \in C(X, [0,1])$ with

$$\chi_i = 1 \text{ on } \overline{V_i^+} \qquad \text{and} \qquad \operatorname{supp}\chi_i \subset B_i.$$

Since the sets sB_i, $s \in S_i$, are pairwise disjoint, so are the sets $s\operatorname{supp}\chi_i$. Define the global perturbation by the cutoff-weighted convex combination

$$q(z) := \left(1 - \sum_i \sum_{s\in S_i} \chi_i(s^{-1}z)\right) q_0(z) + \sum_i \sum_{s\in S_i} \chi_i(s^{-1}z)\widetilde{Q}_i^s(s^{-1}z), \tag{4.15}$$

where each weighted term is extended by zero outside sY_i. The sets sB_i, over all tower labels (i,s), are pairwise disjoint. Hence at every point at most one cutoff is nonzero, and (4.15) is a convex combination of two points of $[0,1]^m$. It is therefore cube-valued. Moreover, $\operatorname{supp}\chi_i \subset B_i$, so every weighted tower term vanishes on a neighbourhood of the boundary of sB_i; its extension by zero is continuous. Hence

$$\|q - q_0\|_\infty < \delta < \varepsilon.$$

Moreover,

$$q^{S_i}(u) = \widetilde{Q}_i(u) \qquad (u \in V_i^+). \tag{4.16}$$

Since $2\delta < \sigma$, the recognition property is preserved:

$$q^F(K_\ell) \cap q^F(X \setminus O_\ell) = \emptyset \qquad (\ell \in \Lambda).$$

Indeed, otherwise equality $q^F(x) = q^F(y)$, with $x \in K_\ell$ and $y \notin O_\ell$, would imply

$$\|q_0^F(x) - q_0^F(y)\|_\infty \le 2\|q - q_0\|_\infty < 2\delta < \sigma,$$

a contradiction. The preserved recognition property and Proposition 4.5(ii) imply that every pair $x, y \in X$ satisfying $q^G(x) = q^G(y)$ lies either in a common tower level $s_0V_i^+$ or in a common remainder chart U_j^+.

In the tower case, suppose $x, y \in s_0 V_i^+$ and denote $u = s_0^{-1} x$ and $v = s_0^{-1} y$. Then $u, v \in V_i^+$. Equality $q^G(x) = q^G(y)$ gives $q(su) = q(sv)$ for every $s \in S_i$ by using the group element ss_0^{-1}. Hence, by (4.16),

$$\widetilde{Q}_i(u) = q^{S_i}(u) = q^{S_i}(v) = \widetilde{Q}_i(v).$$

Choose $C \in \mathcal{U}_i$ with $u \in C$. Since $\widetilde{Q}_i^{-1}(\widetilde{Q}_i(C)) = C$, the equality implies $v \in C$. Refinement of the restricted S_i-join gives sets $W_s \in \mathcal{W}$, $s \in S_i$, such that

$$C \subset Y_i \cap \bigcap_{s \in S_i} s^{-1} W_s.$$

Taking $s = s_0$, we obtain $x = s_0 u, y = s_0 v \in W_{s_0}$, and therefore $\rho(x, y) < \eta$ because $\mathrm{mesh}(\mathcal{W}) < \eta$.

In the second case that $x, y \in U_j^+$, where j is assigned to base i, we denote $u = g_j x$ and $v = g_j y$. Then $u, v \in V_i^+$ and again $\widetilde{Q}_i(u) = \widetilde{Q}_i(v)$. If $\rho(x, y) \geq \eta$, choose $W \in \mathcal{W}$ with $x \in W$. Since $\mathrm{mesh}(\mathcal{W}) < \eta$, we have $y \notin W$. Thus $Y_i \cap g_j(U_j^+ \cap W) \in \mathcal{V}_i$ contains u but not v, contradicting the encoding property. Hence $\rho(x, y) < \eta$ also in this case.

Thus

$$q^G(x) = q^G(y) \quad \Longrightarrow \quad \rho(x, y) < \eta,$$

and so q^G is an η-embedding. $\square$

4.5. **Proof of the main proposition.** We now choose the data in the order required by the preceding results. In particular, the finite set F is chosen before the marker and before the pairs supplied by Proposition 4.5. The observable q_0 is chosen only after those pairs are fixed.

Proof of Proposition 4.1. Fix $f_{\mathrm{orig}} \in C(X, [0,1]^m)$, $\varepsilon > 0$ and $\eta > 0$. We shall construct q such that q^G is an η-embedding and

$$\|q - f_{\mathrm{orig}}\|_\infty < \varepsilon.$$

Choose $\mathrm{mdim}(X, G) < a < \frac{m}{2}$, and let $c := (1/2, \ldots, 1/2)$. For sufficiently small $\lambda > 0$, the map $f_{\mathrm{in}} := (1 - \lambda) f_{\mathrm{orig}} + \lambda c$ takes values in $(0,1)^m$ and satisfies

$$\|f_{\mathrm{in}} - f_{\mathrm{orig}}\|_\infty < \frac{\varepsilon}{4}. \tag{4.17}$$

Choose a finite open cover $\mathcal{W}$ of X with $\mathrm{mesh}(\mathcal{W}) < \eta$. Apply Lemma 3.1 to $\mathcal{W}$, f_{in}, and $\varepsilon/4$. There exist an honest partition structure $\mathcal{H} = \big((\widetilde{O_j})_{j=1}^J, (D_j)_{j=1}^J, \boldsymbol{\psi}\big)$, the vectors $(v_j)_{j=1}^J$, and $f_* = \sum_{j=1}^J v_j \psi_j$. Then

$$\|f_* - f_{\mathrm{in}}\|_\infty < \frac{\varepsilon}{4}. \tag{4.18}$$

Corollary 3.3 provides $K_{\mathcal{H}} \ni e$ and $\delta_{\mathcal{H}} \in (0,1)$.

By amenability, choose a nonempty finite $(K_{\mathcal{H}}, \delta_{\mathcal{H}})$-invariant set A with $e \in A$. Hence Corollary 3.3, applied to A at the point gx, gives

$$\sum_{c \in A} \mathrm{ord}_{\mathcal{D}}(cgx) < a|A| \qquad (g \in G,\ x \in X).$$

Thus the hypothesis of Proposition 4.2 is satisfied. Let $F \ni e$ be the finite window supplied by that proposition.

Choose $\varepsilon_A, \kappa > 0$ so small that

$$c_0 := a(1 + \varepsilon_A) + \kappa < \frac{m}{2}.$$

Fix $\varepsilon_R > 0$. Since G is infinite, choose a nonempty finite set $R \subset G$ such that $\frac{1+\varepsilon_R}{|R|} < \kappa$. By [Nar24, Lemma 3.18], the action has the marker property, as it has URPC. Denote

$$\mathcal{L} := \{e\} \cup R^{-1}R \cup F^{-1}F \cup \mathcal{Q}_F.$$

Choose an $\mathcal{L}$-free sweep-out set U_0. Shrinking U_0 if necessary, choose an open sweep-out set $U \Subset U_0$ which still has finitely many translates covering X. Then the choice of $\mathcal{L}$ ensures that

$$\begin{gathered} (tU)_{t \in F} \text{ is pairwise disjoint,} \\ (rU)_{r \in R} \text{ is pairwise disjoint,} \\ U \cap qU = \emptyset, \qquad q \in \mathcal{Q}_F. \end{gathered} \tag{4.19}$$

Apply Lemma 4.4 to U, and let $\theta > 0$ be the resulting constant and $K_U \subset G$ the resulting finite set. Apply Lemma 2.2 with $E = RF^{-1}$, $K = K_{\mathcal{H}}$, $\delta = \delta_{\mathcal{H}}$, and $\varepsilon_A, \varepsilon_R$. This gives $K_{\mathrm{sp}} \ni e$ and $\delta_{\mathrm{sp}} \in (0, 1)$ such that every sufficiently $(K_{\mathrm{sp}}, \delta_{\mathrm{sp}})$-invariant finite set S satisfies

$$\begin{gathered} |AS| < (1 + \varepsilon_A)|S|, \\ AS \text{ is } (K_{\mathcal{H}}, \delta_{\mathcal{H}})\text{-invariant,} \\ |RF^{-1}S| < (1 + \varepsilon_R)|S|. \end{gathered} \tag{4.20}$$

Set $N_{\mathcal{W}} := \mathrm{mult}(\mathcal{W})$. Choose $\alpha > 0$ so small that

$$\alpha < \min\{\theta, \delta_{\mathrm{sp}}, 1\}, \qquad d_0 := c_0 + N_{\mathcal{W}}\alpha < \frac{m}{2}.$$

Choose d such that

$$d_0 < d < \frac{m}{2}.$$

Finally choose $N \in \mathbb{N}$ so large that, whenever $n \geq N$,

$$1 + \lfloor \alpha n \rfloor \leq \lceil \theta n \rceil, \qquad n > \frac{1}{d - d_0}. \tag{4.21}$$

Choose a finite set $E \subset G$ with $|E| \geq N$, and denote

$$K_* := K_U \cup K_{\mathrm{sp}} \cup E \cup \{e\}.$$

Choose a (K_*, α)-URPC witness

$$\mathfrak{W}^0 = \big(\{S_i\}, \{V_i^0\}, \{g_j\}, \{U_j^0\}\big).$$

Since S_i is (K_*, α)-invariant and $\alpha < 1$, $\left|\bigcap_{k\in K_*} kS_i\right| > (1-\alpha)|S_i| > 0$. Hence we may choose $h_i \in \bigcap_{k\in K_*} kS_i$. Since $E \subset K_*$, for every $k \in E$ we have $h_i \in kS_i$, and therefore $k^{-1}h_i \in S_i$. Thus the map $E \to S_i$, $k \mapsto k^{-1}h_i$, is well defined and injective. Hence

$$|S_i| \geq |E| \geq N.$$

Moreover, $K_U, K_{\mathrm{sp}} \subset K_*$ and $\alpha < \min\{\theta, \delta_{\mathrm{sp}}\}$, so every S_i satisfies the three conditions in (4.20), as well as (4.21). Thus the same witness simultaneously satisfies all shape requirements of Proposition 4.5, Lemma 4.7, and Proposition 4.9.

Apply Proposition 4.5 to this witness. We obtain nested castles and remainder sets together with a finite family of routing pairs

$$K_\ell \subset O_\ell \Subset U, \qquad \ell \in \Lambda,$$

with the closures $\overline{O_\ell}$ pairwise disjoint.

The disjointness conditions on translates of U in (4.19) allow us to apply Proposition 4.2 to this finite family with approximation tolerance $\varepsilon/4$. Choose the resulting localizer $q_0 \in C(X, [0,1]^m)$ and the functions $(\rho_\ell)_{\ell\in\Lambda}$ as in Remark 4.3. Then

$$\|q_0 - f_*\|_\infty < \frac{\varepsilon}{4}, \tag{4.22}$$

and q_0 satisfies the finite-window recognition condition

$$q_0^F(K_\ell) \cap q_0^F(X \setminus O_\ell) = \emptyset \qquad (\ell \in \Lambda).$$

All hypotheses of Proposition 4.9 are now satisfied. Applying that proposition with $\varepsilon/4$, we obtain $q \in C(X, [0,1]^m)$ such that

$$\|q - q_0\|_\infty < \frac{\varepsilon}{4} \tag{4.23}$$

and q^G is an η-embedding.

Combining (4.17), (4.18), (4.22), and (4.23), we obtain

$$\|q - f_{\mathrm{orig}}\|_\infty < \varepsilon.$$

□

5. Proof of Proposition 4.2

The preceding section proves Proposition 4.1, and hence Theorem A, assuming the finite-window localization result in Proposition 4.2. We now prove that proposition. We construct perturbations of the form

$$q_\Theta(z)_j = f_*(z)_j + \langle \Xi(z), \theta^{(j)} \rangle, \qquad 1 \leq j \leq m,$$

where

$$\Theta = (\theta^{(1)}, \dots, \theta^{(m)}) \in (\mathbb{R}^P)^m.$$

The coordinate map $\Xi : X \to \mathbb{R}^P$ is built from translated cutoff functions and polynomial coordinates of the partition-of-unity map. Once Ξ is fixed, choosing Θ sufficiently close to zero makes q_Θ uniformly close to f_*.

This parametrization turns a failure of the finite-window recognition condition into a system of linear equations in the perturbation parameters $\theta^{(j)}$.

We describe the configurations giving rise to such failures using finitely many semialgebraic parameter spaces. By comparing their dimensions with the ranks of the corresponding linear systems, we show that the set of perturbation parameters for which recognition fails has nowhere dense closure. We can therefore choose parameters arbitrarily close to zero for which recognition holds. We begin with the algebraic estimates needed for this argument.

5.1. **Semialgebraic avoidance and polynomial evaluations.** The next lemma is essentially a linear algebra rank counting argument.

Lemma 5.1. *Let Z be a semialgebraic set with* $\dim Z \le D$, *and let*

$$M : Z \longrightarrow \mathrm{Mat}_{n\times P}(\mathbb{R}), \qquad c_\ell : Z \longrightarrow \mathbb{R}^n, \quad 1 \le \ell \le m,$$

be semialgebraic maps. Suppose that $\operatorname{rank} M(z) = r$ *for all* $z \in Z$. *If* $D < mr$, *then the bad parameter set*

$$\mathcal{B} := \left\{ (\theta_1, \dots, \theta_m) \in (\mathbb{R}^P)^m : \begin{array}{l} \textit{there exists } z \in Z \textit{ such that} \\ M(z)\theta_\ell = c_\ell(z) \textit{ for every } 1 \le \ell \le m \end{array} \right\}$$

is semialgebraic and has nowhere dense closure in $(\mathbb{R}^P)^m$.

More generally, let $Z = \bigsqcup_{\alpha=1}^N Z_\alpha$ *be a finite semialgebraic partition such that* M *has constant rank* r_α *on* Z_α. *If* $\dim Z_\alpha < m r_\alpha$ *for* $1 \le \alpha \le N$, *then the same conclusion holds.*

Proof. Consider the set

$$\mathcal{I} := \left\{ (z, \theta_1, \dots, \theta_m) \in Z \times (\mathbb{R}^P)^m : M(z)\theta_\ell = c_\ell(z) \text{ for every } 1 \le \ell \le m \right\},$$

which is semialgebraic.

Let $\pi_Z : \mathcal{I} \to Z$ be the coordinate projection. Fix $z \in Z$. If $\pi_Z^{-1}(z)$ is nonempty, then, for each $1 \le \ell \le m$, the solution set of $M(z)\theta_\ell = c_\ell(z)$ is an affine subspace of $\mathbb{R}^P$ of dimension $P - r$. Consequently,

$$\dim \pi_Z^{-1}(z) = m(P - r).$$

By Lemma 2.9 (v) and the assumption $D < mr$,

$$\begin{aligned} \dim \mathcal{I} &\le \dim \pi_Z(\mathcal{I}) + m(P - r) \le \dim Z + m(P - r) \\ &\le D + m(P - r) < mP. \end{aligned}$$

Let

$$\pi_\theta : Z \times (\mathbb{R}^P)^m \longrightarrow (\mathbb{R}^P)^m$$

be the parameter projection. Then $\mathcal{B} = \pi_\theta(\mathcal{I})$. By Lemma 2.9 (iv), $\mathcal{B}$ is semialgebraic and $\dim \mathcal{B} \le \dim \mathcal{I} < mP$. Lemma 2.9 (vi) implies that $\mathcal{B}$ has nowhere dense closure.

For the final assertion, apply the preceding argument to each Z_α. If $\mathcal{B}_\alpha$ denotes the corresponding bad parameter set, then $\dim \mathcal{B}_\alpha < mP$ for $1 \le \alpha \le N$. Since $\mathcal{B} = \bigcup_{\alpha=1}^N \mathcal{B}_\alpha$, the finite-union property in Lemma 2.9 (i) gives

$$\dim \mathcal{B} = \max_{1 \le \alpha \le N} \dim \mathcal{B}_\alpha < mP.$$

Hence $\mathcal{B}$ again has nowhere dense closure. ∎

We next show that evaluations of sufficiently many monomials at distinct points give linearly independent vectors.

Lemma 5.2. *Let $z_1, \dots, z_N$ be distinct points of Δ^{J-1}. If $d \geq N-1$, then the degree-d Veronese evaluation vectors $\Phi_d(z_1), \dots, \Phi_d(z_N)$ are linearly independent.*

Proof. Suppose that
$$\sum_{i=1}^{N} c_i \Phi_d(z_i) = 0.$$
Fix $j \in \{1, \dots, N\}$. Since $N \leq d+1$, the interpolation property (2.2) provides coefficients $a_{j,\alpha} \in \mathbb{R}$ such that the polynomial
$$P_j(z) := \sum_{\alpha \in \mathcal{A}_{J,d}} a_{j,\alpha} z^\alpha$$
satisfies $P_j(z_i) = \delta_{ij}$ for every i. Taking the scalar product of the assumed vector identity with $a_j := (a_{j,\alpha})_{\alpha \in \mathcal{A}_{J,d}}$, we obtain
$$0 = \sum_{i=1}^{N} c_i \langle a_j, \Phi_d(z_i) \rangle = \sum_{i=1}^{N} c_i P_j(z_i) = c_j.$$
Since j was arbitrary, all coefficients vanish. □

5.2. **Shifted differences and separated marker translates.** The following two lemmas give a rank bound for shifted coordinate differences and a uniqueness statement for visits to translates of a separated set. They will be used when comparing two finite orbit names.

Lemma 5.3. *Let G be a group, let $F \subset G$ be nonempty and finite, let $h \in G \setminus \{e\}$, and let $\beta \in [0,1]$. In $\mathbb{R}^F$, let $(e_t)_{t \in F}$ denote the standard basis and consider*
$$a_s := e_s - \begin{cases} \beta e_{sh}, & sh \in F, \\ 0, & sh \notin F, \end{cases} \qquad s \in F.$$
Then
$$\operatorname{rank}\{a_s : s \in F\} \geq \frac{|F|}{2}.$$
Moreover, if $\beta < 1$, then the vectors a_s, $s \in F$, are linearly independent.

Proof. Consider the directed graph with vertex set F and with an edge
$$s \longrightarrow sh$$
whenever $s, sh \in F$. Since right multiplication by h is injective, every vertex has at most one incoming edge and at most one outgoing edge. Hence every connected component is either a directed path or a directed cycle.

First consider a path
$$s_1 \longrightarrow s_2 \longrightarrow \cdots \longrightarrow s_r,$$

so that $s_i h = s_{i+1}$ for $1 \le i < r$, and $s_r h \notin F$. The corresponding vectors are

$$\begin{aligned} a_{s_1} &= e_{s_1} - \beta e_{s_2}, \\ a_{s_2} &= e_{s_2} - \beta e_{s_3}, \\ &\vdots \\ a_{s_{r-1}} &= e_{s_{r-1}} - \beta e_{s_r}, \\ a_{s_r} &= e_{s_r}. \end{aligned}$$

Relative to the ordered basis $e_{s_1}, \dots, e_{s_r}$, their coefficient matrix is upper triangular with diagonal entries equal to 1. Thus a path with r vertices contributes rank r.

Now consider a directed cycle

$$s_1 \longrightarrow s_2 \longrightarrow \cdots \longrightarrow s_r \longrightarrow s_1.$$

Relative to the same ordering of the vertices, the corresponding matrix is $I_r - \beta P_r$, where

$$P_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

is the cyclic permutation matrix. Since $\det(I_r - \beta P_r) = 1 - \beta^r$, the cycle contributes full rank r whenever $\beta < 1$.

If $\beta = 1$, then $I_r - P_r$ has one-dimensional kernel, spanned by $(1, \dots, 1)$, and therefore has rank $r - 1$. Thus, when $\beta = 1$, each cycle causes a loss of exactly one rank.

Finally, since $h \neq e$, there are no cycles of length one: $sh = s$ would imply $h = e$. Hence every cycle has at least two vertices. If c denotes the number of cycle components, then $c \le \frac{|F|}{2}$. Summing the rank contributions of all path and cycle components gives $\operatorname{rank}\{a_s : s \in F\} \ge |F| - c \ge \frac{|F|}{2}$. When $\beta < 1$, no component loses rank, so the vectors a_s, $s \in F$, are linearly independent. $\square$

Lemma 5.4. *Let $F \subset G$ be finite and denote*

$$\mathcal{Q}_F := \{t_2^{-1} s_2 s_1^{-1} t_1 : s_1, s_2, t_1, t_2 \in F\} \setminus \{e\}.$$

Let $U \subset X$ satisfy $U \cap qU = \emptyset$, $q \in \mathcal{Q}_F$. Let $y \in X$. Suppose that for $i = 1, 2$, there exist $v_i \in U$ and $s_i, t_i \in F$ such that $s_i y = t_i v_i$. Then $v_1 = v_2$ and $s_1^{-1} t_1 = s_2^{-1} t_2$. In particular, if $v_i \in O_{k_i}$, where the sets $(O_k)_{k \in \Lambda}$ are pairwise disjoint subsets of U, then $k_1 = k_2$.

Proof. From $s_1 y = t_1 v_1$, and $s_2 y = t_2 v_2$, we obtain $v_2 = q v_1$, where $q := t_2^{-1} s_2 s_1^{-1} t_1$. If $q \neq e$, then $q \in \mathcal{Q}_F$, while $v_2 = q v_1 \in U \cap qU$, a contradiction. Hence $q = e$, which gives $v_1 = v_2$ and $s_1^{-1} t_1 = s_2^{-1} t_2$. The final assertion follows from the disjointness of the O_k's. $\square$

5.3. **Coordinate values and their dimension bound.** We now record the distinct partition-coordinate values occurring in two orbit names. The notation in this subsection is local to the following lemma.

Let $D_1, \dots, D_J$ be a finite closed cover of X, and let

$$\boldsymbol{\psi} = (\psi_1, \dots, \psi_J) : X \longrightarrow \Delta^{J-1}$$

be a partition of unity satisfying $\operatorname{supp} \psi_j \subset D_j$, $1 \le j \le J$. Write $\mathcal{D} := (D_j)_{j=1}^J$. For $z \in X$, recall $\mathrm{loc}_{\mathcal{D}}(z) = \{j : z \in D_j\}$ and $\mathrm{ord}_{\mathcal{D}}(z) = |\mathrm{loc}_{\mathcal{D}}(z)| - 1$. Then

$$\boldsymbol{\psi}(z) \in \Delta_{\mathrm{loc}_{\mathcal{D}}(z)}. \tag{5.1}$$

Fix $x, y \in X$ and finite sets $A, F \subset G$, with $e \in A$. Denote

$$H := AF, \qquad H^\circ := \bigcap_{c \in A} cF.$$

For $u \in H$, write $p_u := \boldsymbol{\psi}(ux)$ and $q_u := \boldsymbol{\psi}(uy)$.

We call $u \in H$ a **non-loop site** if $p_u \neq q_u$. Let $\mathscr{V}$ be the set of all values among the p_u's and q_u's which occur at a non-loop site, and let

$$\mathscr{V}^\circ := \{v \in \mathscr{V} : v = p_u \text{ or } v = q_u \text{ for some non-loop } u \in H^\circ\}.$$

For $v \in \mathscr{V}$, define

$$I(v) := \bigcap_{\{u : p_u = v\}} \mathrm{loc}_{\mathcal{D}}(ux) \cap \bigcap_{\{u : q_u = v\}} \mathrm{loc}_{\mathcal{D}}(uy),$$

where an empty intersection is omitted, and denote $d(v) := |I(v)| - 1$. Notice that $I(v) \neq \emptyset$, since by (5.1),

$$v \in \bigcap_{\{u : p_u = v\}} \Delta_{\mathrm{loc}_{\mathcal{D}}(ux)} \cap \bigcap_{\{u : q_u = v\}} \Delta_{\mathrm{loc}_{\mathcal{D}}(uy)} = \Delta_{I(v)}.$$

The matrix below records the differences that appear in the collision equations. For each $c \in A$, it assigns a coefficient $+1$ to p_{cs} and -1 to q_{cs} whenever these values belong to $\mathscr{V}^\circ$; if the two values coincide, their contributions cancel. Its rank will provide a lower bound for the rank of the coefficient matrix in the linear equations expressing failure of finite-window recognition.

Finally, define the **core incidence matrix** B°, with rows indexed by $s \in F$ and columns indexed by $(c, v) \in A \times \mathscr{V}^\circ$, by

$$B^\circ_{s,(c,v)} := \mathbf{1}_{\{p_{cs} = v\}} - \mathbf{1}_{\{q_{cs} = v\}},$$

and set $r_\circ := \operatorname{rank} B^\circ$.

A **state** is an element $v \in \mathscr{V}$, that is, a distinct value occurring at a non-loop site. It is a **core state** if $v \in \mathscr{V}^\circ$. The number $d(v) = \dim \Delta_{I(v)}$ is the dimension of its common coordinate face. The next lemma bounds the sum of these dimensions: the core contribution is bounded by the rank of B°, and the remaining contribution by $|H \setminus H^\circ|$.

Lemma 5.5. *Denote* $b_F := |H \setminus H^\circ|$. *Assume that, for every* $s \in F$,

$$\sum_{c\in A} \mathrm{ord}_{\mathcal{D}}(csx) < a|A|, \qquad \sum_{c\in A} \mathrm{ord}_{\mathcal{D}}(csy) < a|A|. \tag{5.2}$$

Then

$$\sum_{v\in\mathscr{V}^\circ} d(v) \le 2ar_\circ \qquad \textit{and} \qquad \sum_{v\in\mathscr{V}\setminus\mathscr{V}^\circ} d(v) \le 2(J-1)b_F.$$

Consequently,

$$\sum_{v\in\mathscr{V}} d(v) \le 2ar_\circ + 2(J-1)b_F.$$

Proof. We first prove that

$$\sum_{v\in\mathscr{V}^\circ} d(v) \le 2ar_\circ. \tag{5.3}$$

Fix $v \in \mathscr{V}^\circ$ and $c \in A$. Choose a non-loop site $u \in H^\circ$ such that $v = p_u$ or $v = q_u$. Since $u \in H^\circ \subset cF$, $s := c^{-1}u \in F$. As $p_u \neq q_u$, exactly one of p_u, q_u equals v. Hence $B^\circ_{s,(c,v)} = \pm 1$. Thus every column of B° is nonzero.

If $r_\circ = 0$, then B° is the zero matrix. This is possible only when $\mathscr{V}^\circ = \emptyset$. Thus, (5.3) holds.

Suppose now that $r_\circ > 0$. Choose $R_0 \subset F$, with $|R_0| = r_\circ$, so that the rows indexed by R_0 form a basis of the row space of B°. For every column (c, v), there exists $s \in R_0$ such that $B^\circ_{s,(c,v)} \neq 0$. Indeed, otherwise that column would vanish on all basis rows, and hence on every row of B°, contradicting the fact that it is nonzero.

For each column (c, v), fix $s(c, v) \in R_0$ such that $B^\circ_{s(c,v),(c,v)} \neq 0$.

For $s \in R_0$ and $c \in A$, denote

$$\mathscr{V}_{s,c} := \{v \in \mathscr{V}^\circ : s(c, v) = s\}.$$

For each fixed $c \in A$, every $v \in \mathscr{V}^\circ$ is assigned exactly one index $s(c, v) \in R_0$. Thus these sets are pairwise disjoint and cover $\mathscr{V}^\circ$, so

$$\mathscr{V}^\circ = \bigsqcup_{s\in R_0} \mathscr{V}_{s,c}. \tag{5.4}$$

By the definition of B°, $B^\circ_{s,(c,v)} = \mathbf{1}_{\{p_{cs}=v\}} - \mathbf{1}_{\{q_{cs}=v\}}$, and for $v \in \mathscr{V}_{s,c}$ this entry is nonzero by the choice of $s(c, v)$. Therefore $\mathscr{V}_{s,c} \subset \{p_{cs}, q_{cs}\}$. Notice also that if $p_{cs} = q_{cs}$, then $\mathscr{V}_{s,c} = \emptyset$.

If $v = p_{cs}$, the occurrence $(cs, +)$ is among those used to define $I(v)$, and hence $I(v) \subset \mathrm{loc}_{\mathcal{D}}(csx)$, and hence $d(v) \le \mathrm{ord}_{\mathcal{D}}(csx)$. Similarly, if $v = q_{cs}$, then $I(v) \subset \mathrm{loc}_{\mathcal{D}}(csy)$ and hence $d(v) \le \mathrm{ord}_{\mathcal{D}}(csy)$. So

$$\sum_{v\in\mathscr{V}_{s,c}} d(v) \le \mathrm{ord}_{\mathcal{D}}(csx) + \mathrm{ord}_{\mathcal{D}}(csy).$$

By (5.4), summing over $c \in A$ and applying the assumed order bounds for each $s \in R_0$, one has

$$\begin{aligned} |A| \sum_{v \in \mathscr{V}^\circ} d(v) &= \sum_{c \in A} \sum_{s \in R_0} \sum_{v \in \mathscr{V}_{s,c}} d(v) \\ &\leq \sum_{s \in R_0} \sum_{c \in A} \big(\mathrm{ord}_{\mathcal{D}}(csx) + \mathrm{ord}_{\mathcal{D}}(csy)\big) \\ &< \sum_{s \in R_0} 2a|A| = 2a|A| r_\circ. \end{aligned}$$

For the strict inequality, apply (5.2) to each $s \in R_0$; the final identity uses $|R_0| = r_\circ$. Dividing by $|A|$ gives

$$\sum_{v \in \mathscr{V}^\circ} d(v) \leq 2ar_\circ.$$

It remains to estimate the values occurring only outside the core. For each $v \in \mathscr{V} \setminus \mathscr{V}^\circ$, choose a non-loop site $u_v \in H \setminus H^\circ$ and one of its two endpoints at which v occurs. This gives an injective map

$$\mathscr{V} \setminus \mathscr{V}^\circ \longrightarrow (H \setminus H^\circ) \times \{+, -\}.$$

Hence

$$|\mathscr{V} \setminus \mathscr{V}^\circ| \leq 2|H \setminus H^\circ|.$$

Since $d(v) \leq J - 1$ for every $v \in \mathscr{V}$, we obtain

$$\sum_{v \in \mathscr{V} \setminus \mathscr{V}^\circ} d(v) \leq 2(J-1)|H \setminus H^\circ|. \qquad \square$$

5.4. **Construction of the perturbation and exclusion of collisions.** We return to the hypotheses of Proposition 4.2. The proof first chooses F and defines the perturbation family. It then describes failed recognition by finitely many types of equations and applies Lemma 5.1.

Proof of Proposition 4.2. Since $a < m/2$, choose $\eta_F > 0$ such that

$$2(J-1)\eta_F < \frac{m-2a}{4}. \tag{5.5}$$

Amenability of G gives a nonempty finite set F with $e \in F$ such that

$$\sum_{c \in A} |cF \triangle F| < \eta_F |F| \qquad \text{and} \qquad |F| > \frac{4}{m-2a}. \tag{5.6}$$

Set

$$H := AF, \qquad H^\circ := \bigcap_{c \in A} cF, \qquad b_F := |H \setminus H^\circ|.$$

Since $e \in A$, one has

$$H \setminus H^\circ \subset \bigcup_{c \in A} (cF \triangle F).$$

Indeed, let $g \in H \setminus H^\circ$. If $g \notin F$, then $g \in cF \setminus F$ for some $c \in A$. If $g \in F$, then $g \notin H^\circ$ gives $g \in F \setminus cF$ for some $c \in A$. In either case $g \in cF \triangle F$. Consequently,

$$b_F \le \sum_{c \in A} |cF \triangle F| < \eta_F |F|.$$

It follows from (5.6) that $b_F < \eta_F|F|$. Together with (5.5) and the lower bound on $|F|$, this gives

$$2(J-1)b_F + 1 < \frac{m-2a}{2}|F|. \tag{5.7}$$

More explicitly, the lower bound on $|F|$ gives $1 < (m-2a)|F|/4$, and therefore

$$2(J-1)b_F + 1 \le 2(J-1)\eta_F|F| + 1 < \frac{m-2a}{4}|F| + 1 < \frac{m-2a}{2}|F|.$$

For each $\ell \in \Lambda$, choose a continuous function $\rho_\ell : X \to [0,1]$ such that

$$\rho_\ell = 1 \text{ on } K_\ell, \qquad \operatorname{supp} \rho_\ell \subset O_\ell. \tag{5.8}$$

For $t \in F$ and $\ell \in \Lambda$, define

$$b_{t,\ell}(z) := \rho_\ell(t^{-1}z), \qquad z \in X,$$

This is continuous.

Choose an integer $d_V \ge 2|H| - 1$, and let $\Phi_{d_V} : \Delta^{J-1} \to \mathbb{R}^{M_{J,d_V}}$ be the Veronese map (see Subsection 2.2). Denote $P := |F||\Lambda| + |A|M_{J,d_V}$ and

$$\Xi : X \to \mathbb{R}^P, \qquad \Xi(z) := \left((b_{t,\ell}(z))_{(t,\ell)\in F\times\Lambda}, (\Phi_{d_V}(\boldsymbol{\psi}(cz)))_{c\in A}\right), \quad z \in X.$$

For $\Theta = (\theta^{(1)}, \ldots, \theta^{(m)}) \in (\mathbb{R}^P)^m$, define

$$q_\Theta(z)_j := f_*(z)_j + \langle \Xi(z), \theta^{(j)} \rangle, \qquad 1 \le j \le m. \tag{5.9}$$

We shall show that the set of Θ for which (4.3) fails has nowhere dense closure.

Fix $\ell \in \Lambda$, $x \in K_\ell$, and $y \in X \setminus O_\ell$. For $s \in F$ and $1 \le j \le m$, (5.9) gives

$$q_\Theta(sx)_j - q_\Theta(sy)_j = f_*(sx)_j - f_*(sy)_j + \langle \Xi(sx) - \Xi(sy), \theta^{(j)} \rangle.$$

Thus a collision gives linear equations in the vectors $\theta^{(j)}$. We now describe their coefficients. For $s \in F$, define the anchor row

$$A_s(x,y) := \big(b_{t,k}(sx) - b_{t,k}(sy)\big)_{(t,k)\in F\times\Lambda}.$$

Since $x \in K_\ell$, (5.8) gives $b_{s,\ell}(sx) = 1$, since $b_{s,\ell}(sx) = \rho_\ell(s^{-1}sx) = \rho_\ell(x)$. If $b_{t,k}(sx) \ne 0$, then $\rho_k(t^{-1}sx) \ne 0$ and hence $sx \in tO_k \subset tU$ and $sx \in sU$. The pairwise disjointness of $(tU)_{t\in F}$ gives $t = s$, and then $x \in O_k \cap O_\ell$. Since the closures of the O_j's are pairwise disjoint, $k = \ell$. Therefore

$$(b_{t,k}(sx))_{(t,k)\in F\times\Lambda} = e_{(s,\ell)}. \tag{5.10}$$

Here $e_{(s,\ell)} \in \mathbb{R}^{F\times\Lambda}$ denotes the standard coordinate vector indexed by (s,ℓ).

Suppose first that $b_{t,k}(sy) = 0$ $s,t \in F$, $k \in \Lambda$. Then $A_s(x,y) = e_{(s,\ell)}$ for every $s \in F$, so these rows are linearly independent. Suppose instead that at least one of the displayed coefficients is nonzero. Choose $s_0, t_0 \in F$ and $k \in \Lambda$ such that $b_{t_0,k}(s_0 y) > 0$, and denote

$$h := s_0^{-1} t_0 \in F^{-1}F, \qquad v := h^{-1} y \in O_k, \qquad \beta := \rho_k(v) \in (0,1]. \tag{5.11}$$

Here $\rho_k(v) = b_{t_0,k}(s_0 y) > 0$ and $s_0 y = t_0 v$. Thus $y = hv$. If $b_{t',k'}(s'y) \neq 0$, then $s'y = t'v'$ for some $v' \in O_{k'}$. By Lemma 5.4 and (4.2),

$$v' = v, \qquad s'^{-1}t' = h, \qquad \text{and} \qquad k' = k. \tag{5.12}$$

Thus the only possible nonzero anchor coefficient at sy is the coordinate (sh, k); it occurs exactly when $sh \in F$, and its value is β. Indeed, (5.12) applies to every nonzero coefficient, so its indices must satisfy $t = sh$ and $k' = k$. When $sh \in F$,

$$b_{sh,k}(sy) = \rho_k\big((sh)^{-1} sy\big) = \rho_k(h^{-1}y) = \rho_k(v) = \beta.$$

Hence

$$A_s(x,y) = e_{(s,\ell)} - \begin{cases} \beta e_{(sh,k)}, & sh \in F, \\ 0, & sh \notin F. \end{cases} \tag{5.13}$$

We now encode, with exact notation, the finitely many discrete records that can be produced by pairs $(x,y) \in K_\ell \times (X \setminus O_\ell)$. Let $\mathscr{E}_H := H \times \{+,-\}$. For such a pair define, for $u \in H$,

$$\xi^{x,y}_{u,+} := \boldsymbol{\psi}(ux), \qquad \xi^{x,y}_{u,-} := \boldsymbol{\psi}(uy),$$

and

$$I^{x,y}_{u,+} := \mathrm{loc}_{\mathcal{D}}(ux), \qquad I^{x,y}_{u,-} := \mathrm{loc}_{\mathcal{D}}(uy).$$

Define an equivalence relation $\sim_{x,y}$ on $\mathscr{E}_H$ by

$$(u,\sigma) \sim_{x,y} (u',\sigma') \quad \Longleftrightarrow \quad \xi^{x,y}_{u,\sigma} = \xi^{x,y}_{u',\sigma'}.$$

The pair also determines a discrete anchor type $\tau_{\ell,x,y}$, which is the symbol $\emptyset$ if all the coefficients $b_{t,k}(sy)$, $s,t \in F$, $k \in \Lambda$, vanish; otherwise (5.12) gives a unique pair $(k,h) \in \Lambda \times F^{-1}F$, and we set $\tau_{\ell,x,y} := (k,h)$. In the latter case $(k,h) \neq (\ell, e)$. Indeed, if $k = \ell$ and $h = e$, then $y = v \in O_\ell$, contrary to the choice of y. Thus, in the subcase $k = \ell$, this condition is simply $h \neq e$. We call $\tau_{\ell,x,y} = \emptyset$ the **no-hit case** and $\tau_{\ell,x,y} = (k,h)$ the **hit case**. Geometrically, $b_{t,k}(sy) > 0$ exactly when $sy \in t\{z \in X : \rho_k(z) > 0\}$.

Define

$$\omega(\ell,x,y) := \big(\ell, (I^{x,y}_{u,\sigma})_{(u,\sigma)\in\mathscr{E}_H}, \sim_{x,y}, \tau_{\ell,x,y}\big)$$

and let

$$\Omega := \{\omega(\ell,x,y) : \ell \in \Lambda,\ x \in K_\ell,\ y \in X \setminus O_\ell\}.$$

The set Ω is finite. Indeed, ℓ ranges over the finite set Λ; each $I^{x,y}_{u,\sigma}$ is one of the finitely many nonempty subsets of $\{1,\dots,J\}$; there are only finitely many equivalence relations on the finite set $\mathscr{E}_H$; and the anchor type belongs to the finite set $\{\emptyset\} \cup \big(\Lambda \times F^{-1}F\big)$.

Fix $\omega \in \Omega$. Write its components as

$$\omega = \big(\ell_\omega, (I^\omega_{u,\sigma})_{(u,\sigma)\in\mathscr{E}_H}, \sim_\omega, \tau_\omega\big),$$

and let $C_\omega(u,\sigma) := [(u,\sigma)]_{\sim_\omega}$ denote the corresponding equivalence class. Since ω is produced by an actual pair, for all $s \in F$

$$\begin{aligned}\sum_{c\in A}(|I^\omega_{cs,+}|-1) &= \sum_{c\in A}\operatorname{ord}_{\mathcal{D}}(csx) < a|A|,\\ \sum_{c\in A}(|I^\omega_{cs,-}|-1) &= \sum_{c\in A}\operatorname{ord}_{\mathcal{D}}(csy) < a|A|.\end{aligned} \tag{5.14}$$

Define the set of non-loop sites of ω by

$$\mathscr{N}_\omega := \{u \in H : C_\omega(u,+) \neq C_\omega(u,-)\}.$$

Let

$$\mathscr{C}_\omega := \{C_\omega(u,\sigma) : u \in \mathscr{N}_\omega,\ \sigma \in \{+,-\}\}$$

be the set of equivalence classes that occur at a non-loop site, and denote

$$\mathscr{C}^\circ_\omega := \{C_\omega(u,\sigma) : u \in \mathscr{N}_\omega \cap H^\circ,\ \sigma \in \{+,-\}\}.$$

For $C \in \mathscr{C}_\omega$, define

$$I_\omega(C) := \bigcap_{(u,\sigma)\in C} I^\omega_{u,\sigma}, \qquad d_\omega(C) := |I_\omega(C)| - 1.$$

The set $I_\omega(C)$ is nonempty. Indeed, choose a pair (x,y) producing ω, and let $\xi^{x,y}_C$ be the common value of $\xi^{x,y}_{u,\sigma}$ for $(u,\sigma) \in C$. Since $\xi^{x,y}_C \in \Delta^{J-1}$, choose j with $(\xi^{x,y}_C)_j > 0$. For every $(u,\sigma) \in C$, membership in $\Delta_{I^\omega_{u,\sigma}}$ implies $j \in I^\omega_{u,\sigma}$. Hence $j \in I_\omega(C)$, proving nonemptiness. Then

$$\xi^{x,y}_C \in \bigcap_{(u,\sigma)\in C} \Delta_{I^\omega_{u,\sigma}} = \Delta_{I_\omega(C)}.$$

Consequently,

$$d_\omega(C) = \dim \Delta_{I_\omega(C)} \geq 0.$$

Define the matrix B°_ω, with rows indexed by $s \in F$ and columns indexed by $(c,C) \in A \times \mathscr{C}^\circ_\omega$, by

$$(B^\circ_\omega)_{s,(c,C)} := \mathbf{1}_{\{C_\omega(cs,+)=C\}} - \mathbf{1}_{\{C_\omega(cs,-)=C\}}, \tag{5.15}$$

and set $r^\circ_\omega := \operatorname{rank} B^\circ_\omega$.

As in the proof of Lemma 5.5, every column of B°_ω is nonzero. Since $A \neq \emptyset$,

$$r^\circ_\omega = 0 \quad\Longleftrightarrow\quad \mathscr{C}^\circ_\omega = \emptyset.$$

Lemma 5.5, together with (5.14), gives

$$\sum_{C\in\mathscr{C}_\omega} d_\omega(C) \leq 2ar^\circ_\omega + 2(J-1)b_F. \tag{5.16}$$

All quantities in this inequality depend only on ω, not on the chosen pair producing it.

We now define a semialgebraic parameter space for the state values associated with the fixed record ω. Let

$$\mathcal{S}_\omega := \left\{ (z_C)_{C\in\mathscr{C}_\omega} \in \prod_{C\in\mathscr{C}_\omega} \Delta_{I_\omega(C)} : z_C \neq z_{C'} \text{ whenever } C \neq C' \right\}. \tag{5.17}$$

If $\mathscr{C}_\omega = \emptyset$, the product in (5.17) is understood to be a singleton.

The space $\mathcal{S}_\omega$ is semialgebraic. Indeed, every $\Delta_{I_\omega(C)}$ is cut out by linear equalities and inequalities, while $z_C \neq z_{C'}$ is equivalent to $\sum_{i=1}^{J}(z_{C,i} - z_{C',i})^2 > 0$. Each coordinate z_C belongs to $\Delta_{I_\omega(C)}$, whose dimension is $d_\omega(C)$. Since $\mathcal{S}_\omega \subseteq \prod_{C\in\mathscr{C}_\omega} \Delta_{I_\omega(C)}$, it follows that (using (5.16))

$$\dim \mathcal{S}_\omega \le \sum_{C\in\mathscr{C}_\omega} d_\omega(C) \le 2ar_\omega^\circ + 2(J-1)b_F. \tag{5.18}$$

Define

$$J_\omega := \begin{cases} \{0\}, & \tau_\omega = \emptyset, \\ (0,1], & \tau_\omega = (k,h) \text{ for some } (k,h), \end{cases}$$

and set

$$Z_\omega := \mathcal{S}_\omega \times J_\omega.$$

A point of Z_ω will be denoted by

$$\zeta = (\mathbf{z}, \beta), \qquad \mathbf{z} = (z_C)_{C\in\mathscr{C}_\omega}.$$

By (5.18),

$$\dim Z_\omega \le 2ar_\omega^\circ + 2(J-1)b_F + \dim J_\omega \le 2ar_\omega^\circ + 2(J-1)b_F + 1. \tag{5.19}$$

Define the affine map

$$\widehat{f}_* : \Delta^{J-1} \longrightarrow \mathbb{R}^m, \qquad \widehat{f}_*(z) := \sum_{i=1}^{J} v_i z_i.$$

Then $f_* = \widehat{f}_* \circ \boldsymbol{\psi}$. For $\zeta = (\mathbf{z}, \beta) \in Z_\omega$, define the anchor matrix $A_\omega(\beta)$, with rows indexed by $s \in F$ and columns indexed by $(t,\nu) \in F \times \Lambda$, as follows. If $\tau_\omega = \emptyset$, set

$$A_\omega(0)_{s,(t,\nu)} := \mathbf{1}_{\{(t,\nu)=(s,\ell_\omega)\}}. \tag{5.20}$$

If $\tau_\omega = (k_\omega, h_\omega)$, set

$$A_\omega(\beta)_{s,(t,\nu)} := \mathbf{1}_{\{(t,\nu)=(s,\ell_\omega)\}} - \beta\, \mathbf{1}_{\{sh_\omega\in F\}} \mathbf{1}_{\{(t,\nu)=(sh_\omega,k_\omega)\}}. \tag{5.21}$$

Define the Veronese matrix $V_\omega(\mathbf{z})$, with rows indexed by $s \in F$ and columns indexed by $(c,\alpha) \in A \times \mathcal{A}_{J,d_V}$, by

$$V_\omega(\mathbf{z})_{s,(c,\alpha)} := \sum_{C\in\mathscr{C}_\omega} \left(\mathbf{1}_{\{C_\omega(cs,+)=C\}} - \mathbf{1}_{\{C_\omega(cs,-)=C\}} \right) z_C^\alpha. \tag{5.22}$$

Finally, define

$$M_\omega(\zeta) := \left[A_\omega(\beta) \mid V_\omega(\mathbf{z}) \right] \in \mathrm{Mat}_{|F|\times P}(\mathbb{R}),$$

and, for $1 \le j \le m$, define $c_{\omega,j} : Z_\omega \to \mathbb{R}^F$ by

$$c_{\omega,j}(\zeta)_s := \sum_{C \in \mathscr{C}_\omega} \big(\mathbf{1}_{\{C_\omega(s,-)=C\}} - \mathbf{1}_{\{C_\omega(s,+)=C\}}\big)\, \widehat{f}_*(z_C)_j. \tag{5.23}$$

Classes occurring only at loop sites are absent from $\mathscr{C}_\omega$ by definition. Moreover, if $C_\omega(u,+) = C_\omega(u,-)$, the two indicators in (5.22) and (5.23) cancel.

The maps M_ω and $c_{\omega,j}$ are semialgebraic. Indeed, (5.20) is constant, (5.21) is affine in β, (5.22) is polynomial in the coordinates of the z_C's, and (5.23) is affine in those coordinates because $\widehat{f}_*$ is affine.

To relate these formulas to an actual pair, let $(x,y) \in K_{\ell_\omega} \times (X \setminus O_{\ell_\omega})$ produce ω. For each $C \in \mathscr{C}_\omega$, let $z_C^{x,y}$ be the common value of $\xi_{u,\sigma}^{x,y}$ for $(u,\sigma) \in C$. These values are pairwise distinct and belong to $\Delta_{I_\omega(C)}$. In the no-hit case denote $\beta^{x,y} := 0$; in a hit case let $\beta^{x,y} = \rho_{k_\omega}(v)$ be defined as in (5.11). Denote

$$\zeta^{x,y} := \big((z_C^{x,y})_{C \in \mathscr{C}_\omega}, \beta^{x,y}\big) \in Z_\omega.$$

For $s \in F$, $c \in A$, and $\alpha \in \mathcal{A}_{J,d_V}$, the defining sum for the Veronese block gives

$$V_\omega(\mathbf{z}^{x,y})_{s,(c,\alpha)} = \boldsymbol{\psi}(csx)^\alpha - \boldsymbol{\psi}(csy)^\alpha.$$

Indeed, if the endpoint classes at cs differ, the two indicators select their respective values; if they agree, both sides vanish. Likewise, the anchor formulas and the reversed order of the indicators in the right-hand side give

$$A_\omega(\beta^{x,y})_{s,(t,\nu)} = b_{t,\nu}(sx) - b_{t,\nu}(sy), \qquad c_{\omega,j}(\zeta^{x,y})_s = f_*(sy)_j - f_*(sx)_j.$$

Here $\mathbf{z}^{x,y} = (z_C^{x,y})_{C \in \mathscr{C}_\omega}$, and the coordinate labeled (c,α) in $\Xi(sx)$ is $\boldsymbol{\psi}(csx)^\alpha$. Hence the two blocks recover the coefficients of the scalar collision equations above.

By (5.10), (5.13), and the definitions of the endpoint classes, the row of $M_\omega(\zeta^{x,y})$ indexed by s is exactly $\Xi(sx) - \Xi(sy)$, while $c_{\omega,j}(\zeta^{x,y})_s = f_*(sy)_j - f_*(sx)_j$. Consequently, the collision equations

$$q_\Theta(sx) = q_\Theta(sy) \qquad (s \in F)$$

are equivalent, in the j-th output coordinate, to

$$M_\omega(\zeta^{x,y})\theta^{(j)} = c_{\omega,j}(\zeta^{x,y}), \qquad 1 \le j \le m. \tag{5.24}$$

To apply Lemma 5.1, we need the parameter-space dimension to be smaller than m times the rank of the collision matrix. We establish two lower bounds for that rank. From (5.20)– (5.21), the anchor rows have exactly the forms already analyzed in (5.13). If $\tau_\omega = \emptyset$, they are the standard basis rows. If $\tau_\omega = (k_\omega, h_\omega)$ and $k_\omega \neq \ell_\omega$, retaining only the columns (t, ℓ_ω), $t \in F$, gives

$$\big(A_\omega(\beta)_{s,(t,\ell_\omega)}\big)_{s,t \in F} = \big(\mathbf{1}_{\{s=t\}}\big)_{s,t \in F} = I_{|F|}.$$

Indeed, none of the negative terms occurs in these columns. Thus these rows are linearly independent. If $k_\omega = \ell_\omega$, then $h_\omega \neq e$, and Lemma 5.3, with e_t

identified with $e_{(t,\ell_\omega)}$, gives rank at least $|F|/2$ for every $\beta \in (0,1]$. Hence, for every $\zeta \in Z_\omega$,

$$\operatorname{rank} A_\omega(\beta) \geq \frac{|F|}{2}, \qquad \operatorname{rank} M_\omega(\zeta) \geq \frac{|F|}{2}. \tag{5.25}$$

Since the rank of M_ω may vary, define its **rank strata** by

$$Z_{\omega,r} := \{\zeta \in Z_\omega : \operatorname{rank} M_\omega(\zeta) = r\}, \qquad 0 \leq r \leq |F|.$$

The rank strata are semialgebraic by the minor conditions for rank, and form a finite partition of Z_ω after empty pieces are omitted. Since Ω is finite, there are at most $|\Omega|(|F|+1)$ nonempty strata in total. On each stratum we must prove $\dim Z_{\omega,r} < mr$. The anchor estimate gives $r \geq |F|/2$; we also need $r \geq r_\omega^\circ$ to control the term $2ar_\omega^\circ$ in (5.19). We obtain this second bound from the Veronese block.

Fix $\mathbf{z} \in \mathcal{S}_\omega$, and suppose that $\lambda = (\lambda_s)_{s\in F} \in \mathbb{R}^F$ satisfies $\lambda^\mathsf{T} V_\omega(\mathbf{z}) = 0$. For every $c \in A$, this gives the vector identity

$$\sum_{C\in\mathscr{C}_\omega} \left(\sum_{s\in F} \lambda_s \big(\mathbf{1}_{\{C_\omega(cs,+)=C\}} - \mathbf{1}_{\{C_\omega(cs,-)=C\}} \big) \right) \Phi_{d_V}(z_C) = 0.$$

There are at most $2|H|$ classes in $\mathscr{C}_\omega$, the points $(z_C)_{C\in\mathscr{C}_\omega}$ are pairwise distinct by (5.17), and $d_V \geq 2|H|-1$. Therefore Lemma 5.2 implies that the vectors $\Phi_{d_V}(z_C)$, $C \in \mathscr{C}_\omega$, are linearly independent. It follows that

$$\sum_{s\in F} \lambda_s \big(\mathbf{1}_{\{C_\omega(cs,+)=C\}} - \mathbf{1}_{\{C_\omega(cs,-)=C\}} \big) = 0$$

for every $c \in A$ and every $C \in \mathscr{C}_\omega$. In particular, using (5.15), $\lambda^\mathsf{T} B_\omega^\circ = 0$. Thus

$$\ker\big(V_\omega(\mathbf{z})^\mathsf{T}\big) \subset \ker\big((B_\omega^\circ)^\mathsf{T}\big).$$

For a $p \times q$ matrix C, the rank–nullity formula gives $\operatorname{rank} C = p - \dim\ker C^\mathsf{T}$. Both matrices above have $|F|$ rows, so

$$\begin{aligned} \operatorname{rank} V_\omega(\mathbf{z}) &= |F| - \dim\ker V_\omega(\mathbf{z})^\mathsf{T} \\ &\geq |F| - \dim\ker (B_\omega^\circ)^\mathsf{T} \\ &= \operatorname{rank} B_\omega^\circ = r_\omega^\circ. \end{aligned} \tag{5.26}$$

Since $M_\omega(\zeta)$ contains both the anchor and Veronese column blocks, if $r := \operatorname{rank} M_\omega(\zeta)$, then (5.25) and (5.26) give

$$r \geq \max\left\{ r_\omega^\circ, \frac{|F|}{2} \right\}. \tag{5.27}$$

Fix a nonempty rank stratum $Z_{\omega,r}$. By (5.27), $r \geq r_\omega^\circ$ and $r \geq |F|/2$. The first inequality controls the main term in (5.19); the second and (5.7) give

$$2ar_\omega^\circ \leq 2ar, \qquad 2(J-1)b_F + 1 < \frac{m-2a}{2}|F| \leq (m-2a)r.$$

Here $a > 0$ follows from (4.1), since the face dimensions are nonnegative, and $m - 2a > 0$. Consequently,

$$\begin{aligned} \dim Z_{\omega,r} &\le 2ar_\omega^\circ + 2(J-1)b_F + 1 \\ &< 2ar + (m-2a)r = mr. \end{aligned} \tag{5.28}$$

For $\omega \in \Omega$ and every nonempty rank stratum $Z_{\omega,r}$, define

$$\mathcal{B}_{\omega,r} := \big\{\Theta \in (\mathbb{R}^P)^m : \exists \zeta \in Z_{\omega,r} \text{ such that} \\ M_\omega(\zeta)\theta^{(j)} = c_{\omega,j}(\zeta)\ \forall 1 \le j \le m\big\}.$$

By Lemma 5.1 and (5.28), each $\mathcal{B}_{\omega,r}$ is semialgebraic and has nowhere dense closure in $(\mathbb{R}^P)^m$. Since Ω is finite and each M_ω has only finitely many possible ranks, the set

$$\mathcal{B} := \bigcup_{\omega\in\Omega} \bigcup_{\substack{0\le r\le|F| \\ Z_{\omega,r}\neq\emptyset}} \mathcal{B}_{\omega,r} \text{ has nowhere dense closure.}$$

Suppose that there are $\ell \in \Lambda$, $x \in K_\ell$, and $y \in X \setminus O_\ell$ such that $q^F_\Theta(x) = q^F_\Theta(y)$. The pair determines the record $\omega = \omega(\ell, x, y) \in \Omega$ and the point $\zeta^{x,y} \in Z_\omega$ constructed above. Let $r = \operatorname{rank} M_\omega(\zeta^{x,y})$. By (5.24), $\Theta \in \mathcal{B}_{\omega,r} \subset \mathcal{B}$. Consequently, if $\Theta \notin \mathcal{B}$, then no such actual collision occurs, and hence

$$q^F_\Theta(K_\ell) \cap q^F_\Theta(X \setminus O_\ell) = \emptyset \qquad (\ell \in \Lambda).$$

Since $\overline{\mathcal{B}}$ has empty interior, every neighbourhood of the origin in $(\mathbb{R}^P)^m$ contains a parameter outside $\mathcal{B}$.

Finally, $f_*(X)$ is a compact subset of $(0,1)^m$, and $\Xi(X)$ is bounded. Hence $q_\Theta \to f_*$ uniformly as $\Theta \to 0$. We may therefore choose $\Theta \notin \mathcal{B}$ sufficiently close to the origin that

$$\|q_\Theta - f_*\|_\infty < \varepsilon \qquad \text{and} \qquad q_\Theta(X) \subset [0,1]^m.$$

Set $q := q_\Theta$. Since $e \in A$, the affine term $f_*(z)$ is determined by $\boldsymbol{\psi}^A(z)$. The formula (5.9) therefore defines a continuous map

$$Q : (\Delta^{J-1})^A \times [0,1]^{F^{-1}\times\Lambda} \longrightarrow \mathbb{R}^m$$

such that

$$q(z) = Q\big(\boldsymbol{\psi}^A(z), \boldsymbol{\rho}^{F^{-1}}(z)\big).$$

Together with (5.8), this also proves the assertions of Remark 4.3. $\square$

References

[Aus88] Joseph Auslander. *Minimal Flows and Their Extensions*, volume 153 of *North-Holland Mathematics Studies*. North-Holland, Amsterdam, 1988.

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real Algebraic Geometry*, volume 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1998.

[BGŚ24] Krzysztof Barański, Yonatan Gutman, and Adam Śpiewak. Prediction of dynamical systems from time-delayed measurements with self-intersections. *Journal de Mathématiques Pures et Appliquées*, 186:103–149, 2024.

[Bil99] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[DL25] Alexander Dranishnikov and Michael Levin. A free $\mathbb{Z}$-action by isometries on a compact metric space which is not embeddable into a cubical shift, 2025. arXiv:2508.14628.

[Dow06] Tomasz Downarowicz. Minimal models for noninvertible and not uniquely ergodic systems. *Israel Journal of Mathematics*, 156:93–110, 2006.

[Eng78] Ryszard Engelking. *Dimension Theory*, volume 19 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1978.

[GL26] Eli Glasner and Chunlin Liu. Comparison and almost finiteness for actions of amenable groups, 2026. arXiv:2609.23702.

[GLM25] Yonatan Gutman, Michael Levin, and Tom Meyerovitch. Equivariant embedding of finite-dimensional dynamical systems. *Mathematische Annalen*, 391:915–936, 2025.

[GLT16] Yonatan Gutman, Elon Lindenstrauss, and Masaki Tsukamoto. Mean dimension of $\mathbb{Z}^k$-actions. *Geometric and Functional Analysis*, 26(3):778–817, 2016.

[GQT19] Yonatan Gutman, Yixiao Qiao, and Masaki Tsukamoto. Application of signal analysis to the embedding problem of $\mathbb{Z}^k$-actions. *Geometric and Functional Analysis*, 29(5):1440–1502, 2019.

[Gro99] Mikhail Gromov. Topological invariants of dynamical systems and spaces of holomorphic maps: I. *Mathematical Physics, Analysis and Geometry*, 2(4):323–415, 1999.

[GS00] Mariano Gasca and Thomas Sauer. Polynomial interpolation in several variables. *Advances in Computational Mathematics*, 12(4):377–410, 2000.

[GT14] Yonatan Gutman and Masaki Tsukamoto. Mean dimension and a sharp embedding theorem: Extensions of aperiodic subshifts. *Ergodic Theory and Dynamical Systems*, 34(6):1888–1896, 2014.

[GT20] Yonatan Gutman and Masaki Tsukamoto. Embedding minimal dynamical systems into Hilbert cubes. *Inventiones Mathematicae*, 221(1):113–166, 2020.

[Gut15] Yonatan Gutman. Mean dimension and Jaworski-type theorems. *Proceedings of the London Mathematical Society*, 111(4):831–850, 2015.

[HW41] Witold Hurewicz and Henry Wallman. *Dimension Theory*, volume 4 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1941.

[Ker20] David Kerr. Dimension, comparison, and almost finiteness. *Journal of the European Mathematical Society*, 22(11):3697–3745, 2020.

[KL16] David Kerr and Hanfeng Li. *Ergodic Theory: Independence and Dichotomies*. Springer Monographs in Mathematics. Springer, Cham, 2016.

[Kri70] Wolfgang Krieger. On entropy and generators of measure-preserving transformations. *Transactions of the American Mathematical Society*, 149:453–464, 1970.

[Kri82] Wolfgang Krieger. On the subsystems of topological Markov chains. *Ergodic Theory and Dynamical Systems*, 2(2):195–202, 1982.

[KS20] David Kerr and Gábor Szabó. Almost finiteness and the small boundary property. *Communications in Mathematical Physics*, 374(1):1–31, 2020.

[Lev23] Michael Levin. Finite-to-one equivariant maps and mean dimension, 2023. arXiv:2312.04689.

[Lin99] Elon Lindenstrauss. Mean dimension, small entropy factors and an embedding theorem. *Publications Mathématiques de l'IHÉS*, 89:227–262, 1999.

[LT14] Elon Lindenstrauss and Masaki Tsukamoto. Mean dimension and an embedding problem: An example. *Israel Journal of Mathematics*, 199(2):573–584, 2014.

[LW00] Elon Lindenstrauss and Benjamin Weiss. Mean topological dimension. *Israel Journal of Mathematics*, 115:1–24, 2000.

[Ma21] Xin Ma. A generalized type semigroup and dynamical comparison. *Ergodic Theory and Dynamical Systems*, 41(7):2148–2165, 2021.

[Mey26] Tom Meyerovitch. A new notion of dimension for dynamical systems and shift embeddability. *Geometric and Functional Analysis*, 36:947–980, 2026.

[Nar24] Petr Naryshkin. URP, comparison, mean dimension, and sharp shift embeddability, 2024. arXiv:2410.01757; updated August 24, 2026.

[Niu22] Zhuang Niu. Comparison radius and mean topological dimension: Rokhlin property, comparison of open sets, and subhomogeneous C^*-algebras. *Journal d'Analyse Mathématique*, 146:595–672, 2022.

[Orn70] Donald Ornstein. Bernoulli shifts with the same entropy are isomorphic. *Advances in Mathematics*, 4(3):337–352, 1970.

[Orn74] Donald S. Ornstein. *Ergodic Theory, Randomness, and Dynamical Systems*, volume 5 of *Yale Mathematical Monographs*. Yale University Press, New Haven, CT, 1974.

[OW87] Donald S. Ornstein and Benjamin Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Analyse Math.*, 48:1–141, 1987.

[Shi26] Ruxi Shi. A sharp embedding theorem for topological dynamical systems, 2026. arXiv:2609.05664.

[SYC91] Tim Sauer, James A. Yorke, and Martin Casdagli. Embedology. *Journal of Statistical Physics*, 65(3–4):579–616, 1991.

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