Δ-WEAK MIXING AND ASYNCHRONOUS LI-YORKE CHAOS

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Abstract. In this paper, we investigate Δ-weakly mixing sets and asynchronous Li–Yorke chaos. First, we prove that in a measure-theoretic sense, every ergodic, non-distal measure preserving system contains many Δ-weakly mixing subsets. Second, we show that every non-trivial transitive system contains an uncountable (r, s)-scrambled set for each pair of distinct $r, s \in \mathbb{N}$; under the stronger assumption of total transitivity, there exists a Mycielski subset that is (r, s)-scrambled simultaneously for all distinct $r, s \in \mathbb{N}$.

1. Introduction

Throughout this paper, a *topological dynamical system* (tds for short) is a pair (X, T), where X is a compact metric space with metric d and $T\colon X \to X$ is continuous. If the set X is infinite, then (X, T) is said to be *non-trivial*. A *measure-preserving system* (mps for short) is a triple (X, μ, T), where (X, T) is a tds and μ is a T-invariant Borel probability measure on X.

We say that a tds (X, T) is *(topologically) transitive* if for every two non-empty open subsets U and V of X, there exists a positive integer n such that $U \cap T^{-n}V \neq \emptyset$. It is *(topologically) weakly mixing* if the product system $(X \times X, T \times T)$ is transitive. A tds (X, T) is Δ-*transitive* if for every $d \geq 2$ there exists a residual subset X_0 of X such that for every $x \in X_0$ the diagonal d-tuple $x^{(d)} := (x, x, \ldots, x)$ has a dense orbit under the action of $T \times T^2 \times \cdots \times T^d$. Glasner [9] showed that for a minimal system, weak mixing implies Δ-transitivity. In [24], Moothathu proved Δ-transitivity implies weak mixing.

Inspired by the intuition underlying the definition of weak mixing, Blanchard and Huang [5] introduced a localized concept termed weakly mixing sets. Building upon this work, Huang, Li, Ye, and Zhou [12] further developed the notion of Δ-weakly mixing sets. A closed subset A of X is Δ-*transitive* if for every $d \geq 2$ there exists a residual subset A_0 of A such that for every $x \in A_0$, the orbit closure of the diagonal d-tuple $x^{(d)} := (x, x, \ldots, x)$ under the action $T \times T^2 \times \cdots \times T^d$ contains A^d. A closed subset A of X with at least two points is Δ-*weakly mixing* if for every $n \geq 1$, A^n is Δ-transitive in the n-th product system $(X^n, T^{(n)})$. Further related results can be found in [16, 23].

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In the first part of this paper, we investigate the existence of Δ-weakly mixing sets. Building upon the results of Blanchard, Glasner, Kolyada, and Maass [4] and employing the methods developed by Blanchard and Huang [5], we know that if a tds admits an ergodic invariant measure that is not measurably distal, then it contains numerous weakly mixing subsets. Huang, Li, Ye, and Zhou [12] strengthened this result by showing the existence of Δ-weakly mixing subsets in the same setting. More precisely, they proved if $\pi : (X, \mu, T) \to (Y, \nu, S)$ is a weakly mixing extension (see Section 2.6 for definition), then there are many Δ-weakly mixing subsets of X. Furthermore, they have the following conjecture [12, Remark 5.7]:

Conjecture 1. *For ν-a.e. $y \in Y$, $\pi^{-1}(y)$ contains a Δ-weakly mixing set.*

We give a positive answer to this conjecture in the following theorem.

Theorem 1.1. *If $\pi : (X, \mu, T) \to (Y, \nu, S)$ is a weakly mixing extension and*

$$\mu = \int \mu_y d\nu(y)$$

is the measure disintegration of μ over ν. Then for ν-a.e. $y \in Y$, $\mathrm{supp}(\mu_y)$ *is a Δ-weakly mixing subset of X.*

Huang, Li, Ye, and Zhou [12] proved that the existence of a Δ-weakly mixing set implies *asynchronous Li–Yorke chaos*; more precisely, there exists an uncountable subset $C \subset X$ such that for any distinct $r, s \in \mathbb{N}$, there exists $\eta = \eta(r, s) > 0$ with

$$\liminf_{n\to\infty} d(T^{rn}x, T^{sn}y) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^{rn}x, T^{sn}y) \geq \eta$$

for all distinct $x, y \in C$. For any $r, s \in \mathbb{N}$, we say that the subset A of X is (r, s)-*scrambled* (with modulus η) if any two distinct points in A satisfy the above two-limit condition. The notion of asynchronous Li–Yorke chaos was introduced by Moothathu [25] as an extension of the classical Li–Yorke chaos due to Li and Yorke [19]. While the classical version may fail to capture multi–time–scale effects, the asynchronous version is designed to address such phenomena. Further work on Li–Yorke chaos and its variants can be found in [1, 2, 4, 6, 10, 11, 13–15, 17, 20–22, 27–30]; see also the survey [18] and references therein.

The second part of this paper is devoted to asynchronous Li–Yorke chaos. In particular, Huang, Li, Ye, and Zhou [12] established asynchronous Li–Yorke chaos for a non-trivial tds (X, T) under several standard hypotheses:

(i) (X, T) has positive topological entropy;
(ii) (X, T) is totally transitive and contains a periodic point;
(iii) (X, T) is scattering;
(iv) (X, T) is weakly mixing.

The second objective of this paper is to establish the pervasive nature of asynchronous Li–Yorke chaos. In particular, cases (ii)-(iv) are direct consequences of the following results.

Theorem 1.2. *Let (X,T) be a non-trivial transitive tds. Then for every pair of distinct $r,s \in \mathbb{N}$, there exists a Mycielski (r,s)-scrambled subset of X (with some modulus $\eta = \eta(r,s) > 0$).*

In the totally transitive case, Theorem 1.2 can be strengthened as follows.

Theorem 1.3. *Let (X,T) be a non-trivial, totally transitive tds. Then there exists a Mycielski subset $C \subset X$ that is (r,s)-scrambled simultaneously for every distinct $r,s \in \mathbb{N}$ (with moduli $\eta = \eta(r,s) > 0$).*

This paper is organized as follows. In Section 2, we review some notions and required results. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2 and Theorem 1.3.

2. Preliminaries

In this section we recall some basic definitions and results.

2.1. **Density of subsets of non-negative integers.** Let $\mathbb{N}$ denote the set of natural numbers, and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For a subset $F \subset \mathbb{Z}_+$, the *upper density* of F is

$$\overline{D}(F) := \limsup_{n\to\infty} \frac{|F \cap \{0,1,\dots,n-1\}|}{n},$$

where $|\cdot|$ denotes the cardinality function for finite sets. Similarly, the *lower density* of F is

$$\underline{D}(F) := \liminf_{n\to\infty} \frac{|F \cap \{0,1,\dots,n-1\}|}{n}.$$

We say that F has *density* $D(F)$ if $\underline{D}(F) = \overline{D}(F)$, in which case $D(F)$ denotes this common value.

2.2. **Compact metric spaces and hyperspaces.** Let (X,d) be a compact metric space. We say that a non-empty subset A of X is *totally disconnected* if its only connected subsets are singletons. By a *perfect* set, we mean a non-empty closed set without isolated point, by a *Cantor* set, we mean a compact, perfect and totally disconnected set. We write $B(x,\epsilon)$ (resp. $\overline{B}(x,\epsilon)$) for the open (resp. closed) ball of radius ϵ centered at x. A subset $K \subset X$ is called a *Mycielski* set if it is a union of countably many Cantor sets, which was introduced in [4]. For $n \geq 2$, let $\Delta^{(n)} = \{(x_1,x_2,\dots,x_n) \in X^n : x_i = x_j \text{ for some } 1 \leq i < j \leq n\}$. For $A \subset X$, write $A^n := A \times \cdots \times A (n \text{ times})$. The following result is from [26, Theorem 1] which we shall use.

Theorem 2.1. *Suppose that X is a compact and perfect metric space. If $n \geq 2$ and R_n is a dense G_δ subset of X^n, then there exists a Mycielski subset K of X such that $K^n \subset R_n \cup \Delta^{(n)}$.*

Let 2^X denote the hyperspace of all non-empty closed subsets of X. We equip 2^X with the Hausdorff metric

$$d_H(A,B) := \max\left\{ \sup_{x\in A}\inf_{y\in B} d(x,y),\ \sup_{y\in B}\inf_{x\in A} d(x,y)\right\},\ \text{for } A,B\in 2^X.$$

It is well-known that if X is compact, then $(2^X, d_H)$ is compact.

2.3. **Topological dynamics.** Let (X,T) be a tds. For two subsets U and V of X, define the *hitting time set* of U and V by

$$N(U,V) := \{n\in\mathbb{N} : U\cap T^{-n}V \neq \emptyset\}.$$

A tds (X,T) is *transitive* if for any two non-empty open subset U and V, the set $N(U,V)\neq\emptyset$. A tds (X,T) is *totally transitive* if for any $r\in\mathbb{N}$, (X,T^r) is transitive. Note that if a tds (X,T) is transitive, then either it is a periodic orbit or perfect. In particular, if it is non-trivial, then X is perfect.

A point $x\in X$ is *transitive* if its orbit $\{\,T^n x : n\geq 0\,\}$ is dense in X. Denote by $\text{Trans}(X,T)$ the set of all transitive points of (X,T). It is well-known that if (X,T) is transitive, then $\text{Trans}(X,T)$ is a dense G_δ subset of X.

2.4. **Δ-weakly mixing.** Let (X,T) be a tds. A closed subset $E\subset X$ is called Δ-*transitive* if there exists a residual subset $A\subset E$ such that for every $x\in A$ and every $d\in\mathbb{N}$,

$$E^d \subset \overline{\left\{(T^n x,\, T^{2n}x,\, \ldots,\, T^{dn}x) :\, n\in\mathbb{N}\right\}}.$$

A closed subset $E\subset X$ is called Δ-*weakly mixing* if E^m is Δ-transitive in the product system $(X^m, T^{(m)})$ for every $m\in\mathbb{N}$, where $T^{(m)} := T\times\cdots\times T(m \text{ times})$.

For $\ell\in\mathbb{N}$ and $U, U_1, U_2, \ldots, U_\ell\subset X$, the *hitting time set* of $U, U_1, U_2, \ldots, U_\ell$ is defined by

$$N\left(U; U_1, U_2, \ldots, U_\ell\right) := \left\{k\in\mathbb{N} : U\cap\bigcap_{i=1}^{\ell} T^{-ik}U_i \neq\emptyset\right\}.$$

The following result is a characterization of Δ-transitive sets (see [12, Proposition 3.3]).

Proposition 2.2. *Let (X,T) be a tds. A closed subset E of X is Δ-transitive if and only if for every $\ell\in\mathbb{N}$ and any non-empty open subsets $U_0, U_1, U_2, \ldots, U_\ell$ of X intersecting E,*

$$N\left(U_0\cap E; U_1, U_2, \ldots, U_\ell\right)\neq\emptyset.$$

By Proposition 2.2, we have the following characterization of Δ-weakly mixing sets.

Proposition 2.3. *Let (X,T) be a tds. A closed subset E of X is Δ-weakly mixing if and only if for any $m,\ell\in\mathbb{N}$ and any non-empty open subsets $U_i, i\in\{1,2,\ldots,m\}$ of X intersecting E, one has that*

$$\bigcap_{i_0,i_1,\cdots,i_\ell\in\{1,2,\ldots,m\}} N\left(U_{i_0}\cap E; U_{i_1}, U_{i_2}\ldots, U_{i_\ell}\right)\neq\emptyset.$$

2.5. **Measure disintegration and relative product.** Let (X,d) be a compact metric space and let $\mathcal{X}$ denote its Borel σ-algebra. Write $\mathcal{M}(X)$ for the set of Borel probability measures on X. We endow $\mathcal{M}(X)$ with the weak*-topology; that is, $\lim_{n\to\infty}\mu_n=\mu$ if and only if for any continuous function f on X,

$$\lim_{n\to\infty}\int_X f\,d\mu_n=\int_X f\,d\mu.$$

For $\mu\in\mathcal{M}(X)$, we denote by $\mathrm{supp}(\mu)$ the topological support of μ.

For $\mu\in\mathcal{M}(X)$ and a sub-σ-algebra $\mathcal{C}$ of $\mathcal{X}$, μ can be disintegrated over $\mathcal{C}$ as

$$\mu=\int_X \mu_{\mathcal{C},x}d\mu(x),$$

where $\mu_{\mathcal{C},x}\in\mathcal{M}(X)$ such that for any $f\in L^1(X,\mu)$, the function $x\mapsto\displaystyle\int_X fd\mu_{\mathcal{C},x}$ is $\mathcal{C}$-measurable. Moreover,

$$\mathbb{E}^{\mu}(f|\mathcal{C})(x)=\int_X fd\mu_{\mathcal{C},x}\text{ for }\mu\text{-a.e. }x\in X,$$

where $\mathbb{E}^{\mu}(f|\mathcal{C})$ is the conditional expectation of f over the σ-algebra $\mathcal{C}$ with respect to the measure μ. Especially, for μ-a.e. $x\in X$,

$$\mu_{\mathcal{C},x}=\mu_{\mathcal{C},x'}\text{ for }\mu_{\mathcal{C},x}\text{-a.e. }x'\in X.$$

For a measurable map $\pi:(X,\mathcal{X},\mu)\to(Y,\mathcal{Y},\nu)$ with $\mu\circ\pi^{-1}=\nu$, there is a natural disintegration of $\mu=\displaystyle\int_Y\mu_y d\nu(y)$, called *the disintegration of* μ *over* $(Y,\mathcal{Y},\nu)$, such that

$$\mu_{\pi(x)}=\mu_{\pi^{-1}(\mathcal{Y}),x}\text{ for }\mu\text{-a.e. }x\in X.$$

For $i=1,2$, let $\pi_i:(X_i,\mathcal{X}_i,\mu_i)\to(Y,\mathcal{Y},\nu)$ be measurable maps and let

$$\mu_i=\int_Y\mu_{i,y}\,d\nu(y)$$

be the disintegrations of μ_i over $(Y,\mathcal{Y},\nu)$. Denote by $\mu_1\times_Y\mu_2$ the measure on $X_1\times X_2$ defined by

$$\mu_1\times_Y\mu_2=\int_Y(\mu_{1,y}\times\mu_{2,y})d\nu(y).$$

The measure space $(X_1\times X_2,\mathcal{X}_1\otimes\mathcal{X}_2,\mu_1\times_Y\mu_2)$ is called the product of $(X_1,\mathcal{X}_1,\mu_1)$ and $(X_2,\mathcal{X}_2,\mu_2)$ *relative to* $(Y,\mathcal{Y},\nu)$.

2.6. **Measure preserving systems and weakly mixing extension.** The following multiple recurrence theorem (Cf. [8, Theorem A]) is used in the proof of Theorem 1.1.

Theorem 2.4. *If $T_1, T_2, \ldots, T_\ell$ are commuting measure preserving transformations of a probability measure space $(X, \mathcal{X}, \mu)$ and $A \in \mathcal{X}$ with $\mu(A) > 0$, then*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T_1^{-n}A \cap T_2^{-n}A \cap \cdots \cap T_\ell^{-n}A) > 0.$$

Let $\pi : (X, \mu, T) \to (Y, \nu, S)$ be a factor map, that is, $\pi : X \to Y$ is measurable, $\pi \circ T = S \circ \pi$, and $\mu \circ \pi^{-1} = \nu$. We call the extension π *ergodic* if every T-invariant measurable set $A \in \mathcal{X}$ satisfies $A = \pi^{-1}(B)$ modulo μ-null sets for some S-invariant measurable set $B \in \mathcal{Y}$.

Let $\mu = \int_Y \mu_y \, d\nu(y)$ be the disintegration of μ over $(Y, \mathcal{Y}, \nu)$ and set

$$\mu \times_Y \mu \ := \ \int_Y (\mu_y \times \mu_y) \, d\nu(y)$$

the relatively independent self-joining of μ over Y. Consider the system $(X \times X, \mu \times_Y \mu, T \times T)$ together with the factor map

$$\psi = \pi \circ \pi_1 : X \times X \to Y, \qquad \psi(x, x') = \pi(x),$$

where π_1 denotes the projection onto the first coordinate. We say that π is a *weakly mixing extension* if ψ is ergodic.

We record the following fiberwise mixing statement, which is restated from [7, Lemma 7.8] in a form tailored to our analysis. This formulation will serve as a key technical tool in subsequent proofs.

Proposition 2.5. *Let $\pi : (X, \mu, T) \to (Y, \nu, S)$ be a weakly mixing extension, and let $\mu = \int_Y \mu_y \, d\nu(y)$ be the disintegration of μ over $(Y, \mathcal{Y}, \nu)$. Fix $L \in \mathbb{N}$ and $f_0, \ldots, f_L \in L^\infty(X, \mu)$. Then there exists a subset $H \subset \mathbb{N}$ with density 1 such that*

$$\lim_{\substack{n\to\infty \\ n\in H}} \int_Y \left| \int_X \prod_{i=0}^{L} f_i \circ T^{in} \, d\mu_y \ - \ \prod_{i=0}^{L} \int_X f_i \circ T^{in} \, d\mu_y \right| d\nu(y) = 0.$$

3. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. We begin with a technical lemma that constitutes a fundamental component of our argumentation.

Lemma 3.1. *Let $\pi : (X, \mu, T) \to (Y, \nu, S)$ be a weakly mixing extension and let $\mu = \int_Y \mu_y \, d\nu(y)$ be the disintegration of μ over $(Y, \mathcal{Y}, \nu)$. Let $D \subset Y$ be a compact subset with $\nu(D) > 0$. Let $L \in \mathbb{N}$. Then for any $\eta > 0$, there exists a compact subset $D^* \subset D$ with $\nu(D^*) > 0$ and a finite subset $E \subset X$ such that*

(1) for $y \in D^*$, $\bigcup_{x\in E}(B(x, \eta) \cap supp(\mu_y))$ is η-dense in $supp(\mu_y)$;

(2) for $y \in D^*$*, there is* $n \in \mathbb{N}$ *such that*

$$\mathit{supp}(\mu_y) \cap \left(\bigcap_{l=0}^{L} T^{-nl} B(x_l, \eta) \right) \neq \emptyset \text{ for all } x_0, x_1, \dots, x_L \in E.$$

Proof. Note that the maps

$$Y \ni y \mapsto \mu_y \in \mathcal{M}(X) \quad \text{(with the weak}^* \text{ topology)}$$

and

$$Y \ni y \mapsto \operatorname{supp}(\mu_y) \in 2^X \quad \text{(with the Hausdorff metric)}$$

are Borel measurable. Hence, by Lusin's theorem, we can choose a compact subset $D' \subset D$ with $\nu(D') > 0$ such that the maps $y \mapsto \mu_y$ and $y \mapsto \operatorname{supp}(\mu_y)$ are continuous on D'.

Fix $y^* \in \operatorname{supp}(\nu|_{D'})$, $\eta > 0$ and a $\frac{\eta}{2}$-dense finite subset E of $\operatorname{supp}(\mu_{y^*})$. The continuity of $y \mapsto \operatorname{supp}(\mu_y)$ and $y \mapsto \mu_y$ on D' ensures that there exist $\epsilon > 0$ and $\delta > 0$ such that

(i) for $y \in D' \cap \overline{B}(y^*, \epsilon)$, $\cup_{x \in E}(B(x, \eta) \cap \operatorname{supp}(\mu_y))$ is η-dense in $\operatorname{supp}(\mu_y)$;
(ii) for $y \in D' \cap \overline{B}(y^*, \epsilon)$, $\mu_y(B(x, \eta)) \geq \delta$ for any $x \in E$.

Let

$$D'' = D' \cap \overline{B}(y^*, \epsilon).$$

Consequently, $\nu(D'') > 0$, since $y^* \in \operatorname{supp}(\nu|_{D'})$. For convenience, denote $C_x := B(x, \eta)$ for $x \in E$, and write $\mathfrak{C} := \{C_x : x \in E\}$. By Proposition 2.5, for every $L \in \mathbb{N}$ and any finite sequence $C_0, \dots, C_L \in \mathfrak{C}$, there exists a subset $H(C_0, \dots, C_L) \subset \mathbb{N}$ of density 1 such that

$$\lim_{\substack{n \to \infty \\ n \in H(C_0, \dots, C_L)}} \int_Y \left| \mu_y \left(\bigcap_{l=0}^{L} T^{-nl} C_l \right) - \prod_{l=0}^{L} \mu_y(T^{-nl} C_l) \right| d\nu(y) = 0.$$

In particular,

$$\lim_{\substack{n \to \infty \\ n \in H(C_0, \dots, C_L)}} \int_{\bigcap_{0 \leq l \leq L} T^{-nl} D''} \left| \mu_y \left(\bigcap_{l=0}^{L} T^{-nl} C_l \right) - \prod_{l=0}^{L} \mu_y(T^{-nl} C_l) \right| d\nu(y) = 0. \tag{3.1}$$

Since $\mathfrak{C}$ is finite, the intersection

$$H := \bigcap_{C_0, \dots, C_L \in \mathfrak{C}} H(C_0, \dots, C_L)$$

also has density 1. By Theorem 2.4, there exist $c > 0$ and a positive density subset H' of $\mathbb{N}$ such that for $n \in H'$,

$$\nu \left(\bigcap_{l=0}^{L} T^{-nl} D'' \right) > c.$$

It is clear that $H' \cap H$ has positive density. Due to (3.1), we can find $n \in H' \cap H$ such that for $C_0, \dots, C_L \in \mathfrak{C}$

$$\int_{\bigcap_{0\le l\le L} T^{-nl}D''} \left| \mu_y\left(\bigcap_{l=0}^{L} T^{-nl}C_l\right) - \prod_{l=0}^{L} \mu_y(T^{-nl}C_l) \right| d\nu(y) < \frac{c\,\delta^{L+1}}{2\,|E|^{L+1}}.$$

For $C_0, \dots, C_L \in \mathfrak{C}$, put

$$D(C_0, \dots, C_L) := \left\{ y \in \bigcap_{l=0}^{L} T^{-nl}D'' : \left| \mu_y\left(\bigcap_{l=0}^{L} T^{-nl}C_l\right) - \prod_{l=0}^{L} \mu_y(T^{-nl}C_l) \right| < \delta^{L+1} \right\}.$$

Then, for any finite sequence $C_0, \dots, C_L \in \mathfrak{C}$,

$$\nu(D(C_0, \dots, C_L)) \ge \nu\left(\bigcap_{l=0}^{L} T^{-nl}D''\right) - \frac{c}{2|E|^{L+1}},$$

which implies that

$$\nu\left(\bigcap_{C_0,\dots,C_L\in\mathfrak{C}} D(C_0, \dots, C_L)\right) > 0. \tag{3.2}$$

Note that for $y \in \bigcap_{l=0}^{L} T^{-nl}D''$, we have that $y, T^n y, \dots, T^{nL}y \in D''$, which, together with (ii), implies that

$$\prod_{l=0}^{L} \mu_y(T^{-nl}C_l) \ge \delta^{L+1}.$$

Hence, for any $y \in D(C_0, \dots, C_L)$,

$$\mu_y\left(\bigcap_{l=0}^{L} T^{-nl}C_l\right) > 0.$$

Combining this with (3.2), there exists a compact subset

$$D^* \subset \bigcap_{C_0,\dots,C_L\in\mathfrak{C}} D(C_0, \dots, C_L) \subset D''$$

with $\nu(D^*) > 0$ such that for any $y \in D^*$ and $C_0, \dots, C_L \in \mathfrak{C}$,

$$\mu_y\left(\bigcap_{l=0}^{L} T^{-nl}C_l\right) > 0.$$

Hence, for any $L \in \mathbb{N}$ and $x_0, x_1, \dots, x_L \in E$,

$$\mathrm{supp}(\mu_y) \ \cap \ \left(\bigcap_{l=0}^{L} T^{-nl}B(x_l, \eta)\right) \ne \emptyset.$$

Thus, item (2) in this lemma is satisfied and by (i) and the fact that $D^* \subset D''$, item (1) in this lemma also holds. This completes the proof of the lemma. □

Proof of Theorem 1.1. Let $\pi : (X, \mu, T) \to (Y, \nu, S)$ be a weakly mixing extension and let $\mu = \int_Y \mu_y \, d\nu(y)$ be the disintegration of μ over $(Y, \mathcal{Y}, \nu)$. For $l \in \mathbb{N}$, define $W_l \subset Y$ to be the set of points y for which there exist $M \in \mathbb{N}$ and non-empty open sets $U_1, \ldots, U_M \subset X$ intersecting $\mathrm{supp}(\mu_y)$ such that

$$\bigcap_{m_0, m_1, \ldots, m_l \in \{1, \ldots, M\}} N(U_{m_0} \cap \mathrm{supp}(\mu_y)\,;\, U_{m_1}, \ldots, U_{m_l}) = \emptyset.$$

By Proposition 2.3, to prove Theorem 1.1, it suffices to demonstrate that W_l is ν-null for every $l \in \mathbb{N}$. Suppose, toward a contradiction, that $\nu(W_k) > 0$ for some fixed $k \in \mathbb{N}$.

Since $\nu(W_k) > 0$, iterating Lemma 3.1 for $\eta = 2^{-i}, i \in \mathbb{Z}_+$ yields a decreasing sequence $\{D_i\}_{i \in \mathbb{Z}_+}$ of non-empty compact subsets of W_k (i.e., $W_k \supset D_0 \supset D_1 \supset D_2 \supset \cdots$), with the following properties: for each $i \in \mathbb{Z}_+$, there exists a finite set $E_i \subset X$ such that the following hold:

(P1) For every $y \in D_i$,

$$\Big(\bigcup_{x \in E_i} B(x, 2^{-i}) \Big) \cap \mathrm{supp}(\mu_y) \;\text{ is }\; 2^{-i}\text{-dense in } \mathrm{supp}(\mu_y).$$

(P2) For every $y \in D_i$, there exists $n_i \in \mathbb{N}$ such that for any $x_0, x_1, \ldots, x_L \in E_i$,

$$\mathrm{supp}(\mu_y) \cap \left(\bigcap_{l=0}^{k} T^{-n_i l} B(x_l, 2^{-i}) \right) \neq \emptyset.$$

Since $\{D_i\}_{i \in \mathbb{Z}_+}$ is a decreasing sequence of non-empty compact sets, $\bigcap_{i \in \mathbb{Z}_+} D_i \neq \emptyset$. Fix $y \in \bigcap_{i \in \mathbb{Z}_+} D_i$. We now show that this leads to a contradiction with the definition of W_k.

Given $M \in \mathbb{N}$ and non-empty open sets $U_0, U_1, \ldots, U_M \subset X$ intersecting $\mathrm{supp}(\mu_y)$, we choose $j \in \mathbb{Z}_+$ large enough such that each of $U_0, U_1, \ldots, U_M$ contains a ball of diameter strictly larger than $4 \cdot 2^{-j}$, whose center lies in $\mathrm{supp}(\mu_y)$. Then by property (P1), for each of $U_0, U_1, \ldots, U_M$, we can find $x_m \in E_j$ such that

$$B(x_m, 2^{-j}) \subset U_m, m = 0, 1, \ldots, M.$$

Then we use property (P2) to obtain that

$$\mathrm{supp}(\mu_y) \;\cap\; \bigcap_{l=0}^{k} T^{-n_j l} U_{m_l} \;\neq\; \emptyset \quad \text{for all } \; 0 \le m_0, m_1, \ldots, m_k \le M.$$

Hence,

$$\bigcap_{m_0, m_1, \ldots, m_k \in \{0, 1, \ldots, M\}} N(U_{m_0} \cap \mathrm{supp}(\mu_y)\,;\, U_{m_1}, \ldots, U_{m_k}) \neq \emptyset.$$

Clearly, this contradicts with the fact that $y \in W_k$. This completes the proof of Theorem 1.1. □

4. PROOFS OF THEOREM 1.2 AND 1.3

In this section, we prove Theorem 1.2 and 1.3. Recall that if (X,T) is transitive, then $\mathrm{Trans}(X,T)$ is a dense G_δ subset of X (see subsection 2.3 for details). Our arguments depend on the following decomposition of a transitive tds as established in [3].

Proposition 4.1. *Let (X,T) be a transitive tds and $r \in \mathbb{N}$. Then there exist a unique divisor k of r and unique closed subsets $X_0,\dots,X_{k-1}$ of X such that*

(1) $T(X_i)=X_{i+1}$ for $i=0,\dots,k-1$;
(2) (X_i,T^r) is transitive for each $i=0,\dots,k-1$;
(3) $X=\bigcup_{i=0}^{k-1}X_i$;
(4) $\mathrm{int}(X_i)\cap\mathrm{int}(X_j)=\emptyset$ whenever $0\le i\ne j\le k-1$;
(5) $\overline{\mathrm{int}(X_i)}=X_i$ for each $i=0,\dots,k-1$.

For $r,s\in\mathbb{N}$ and $\eta>0$, we put

$$\mathrm{W}_{r,s}(\eta)=\{(x,y)\in X\times X:\limsup_{n\to\infty}d(T^{rn}x,T^{sn}y)>\eta\},$$

and

$$\mathrm{Prox}(r,s)=\{(x,y)\in X\times X:\liminf_{n\to\infty}d(T^{rn}x,T^{sn}y)=0\}.$$

It is clear that both $\mathrm{W}_{r,s}(\eta)$ and $\mathrm{Prox}(r,s)$ are G_δ subsets of $X\times X$. We now apply the above decomposition to derive two lemmas. These lemmas show that for distinct $r,s\in\mathbb{N}$, $\mathrm{W}_{r,s}(\eta)$ and $\mathrm{Prox}(r,s)$ are “abundant” in transitive systems. Taken together, they imply Theorem 1.2 and 1.3.

Lemma 4.2. *Let (X,T) be a non-trivial transitive tds. Fix distinct $r,s\in\mathbb{N}$, and let $X_0,\dots,X_{k-1}$ be the closed subsets obtained by Proposition 4.1 for r. Then there exists $\eta>0$ such that*

$$\left(\bigcup_{i=0}^{k-1}\mathrm{Trans}(X_i,T^r)\right)\times X\subset W_{r,s}(\eta).$$

Proof. For each $i\in\{0,\dots,k-1\}$, we show that there exists $\eta_i>0$ such that

$$\mathrm{Trans}(X_i,T^r)\times X\subset W_{r,s}(\eta_i).$$

Set $\eta:=\min_{0\le i\le k-1}\eta_i>0$. Then it follows

$$\Big(\bigcup_{i=0}^{k-1}\mathrm{Trans}(X_i,T^r)\Big)\times X\subset W_{r,s}(\eta).$$

Consequently, it suffices to consider only the case $i=0$, with all other cases following identically. Since (X,T) is non-trivial, so is (X_0,T^r). Hence, for every transitive point $y \in \mathrm{Trans}(X_0, T^r)$, its orbit under T is infinite; in particular, we have $T^m y \neq T^n y$ whenever $m \neq n$. We now fix $y \in \mathrm{Trans}(X_0, T^r)$, and set

$$\delta := \tfrac{1}{2}\, d(T^r y, T^s y) > 0.$$

By the continuity of T^r and T^s, choose $\rho > 0$ sufficiently small such that whenever $d(u, y) < \rho$ and $d(v, y) < \rho$ one has

$$d\big(T^r u,\, T^s v\big) \;\geq\; \delta. \tag{4.1}$$

Fix $y' \in \mathrm{Trans}(X_0, T^r)$. Now we start to prove that there exists $\eta_0 > 0$ such that for any $z \in X$, $(y', z) \in W_{r,s}(\eta_0)$. Indeed, the transitivity ensures there exists a strictly increasing infinite sequence $\{n_j\}_{j=1}^{\infty}$ with

$$\lim_{j\to\infty} d\big(T^{rn_j} y',\, y\big) = 0. \tag{4.2}$$

Regardless of the choice of point $z \in X$, at least one case below holds:

Case 1. There exist infinitely many indices j satisfy $d(T^{sn_j} z, y) \geq \rho$. Thus, applying (4.2), for all sufficiently large such j, we have

$$d\big(T^{rn_j} y',\, T^{sn_j} z\big) \;\geq\; d\big(T^{sn_j} z,\, y\big) - d\big(T^{rn_j} y',\, y\big) \;>\; \rho/2.$$

Hence, $\limsup_{n\to\infty} d(T^{rn} y', T^{sn} z) \geq \rho/2$.

Case 2. There exist infinitely many indices j satisfy $d(T^{sn_j} z, y) < \rho$. For all sufficiently large such j, we derive from (4.2) that $d(T^{rn_j} y', y) < \rho$, and thus (4.1) yields

$$d\big(T^{r(n_j+1)} y',\, T^{s(n_j+1)} z\big) \;\geq\; \delta.$$

Hence $\limsup_{n\to\infty} d(T^{rn} y', T^{sn} z) \geq \delta$.

Let $\eta_0 = \min\{\rho/2, \delta\} > 0$. Combining the above two cases, we find that for every $z \in X$, $(y', z) \in W_{r,s}(\eta_0)$. Therefore,

$$\mathrm{Trans}(X_0, T^r) \times X \subset W_{r,s}(\eta_0).$$

The proof is completed. □

Lemma 4.3. *Let (X, T) be a non-trivial transitive tds. Fix distinct $r, s \in \mathbb{N}$, and let $X'_0, X'_1, \ldots, X'_{k'-1}$ be the closed subsets obtained by Proposition 4.1 for $|r - s|$. Then for any $i = 0, 1, \ldots, k' - 1$, $\mathrm{Prox}(r, s) \cap (X'_i \times X'_i)$ is a dense G_δ subset of $X'_i \times X'_i$.*

Proof. Without loss of generality, assume that $r > s$. The argument for $r < s$ proceeds analogously. By the uniform continuity of T, for any $n \in \mathbb{N}$, there exists $\eta_n > 0$ such that for all $x, y \in X$,

$$d(x, y) < \eta_n \implies d\big(T^{sn} x, T^{sn} y\big) < \frac{1}{n}. \tag{4.3}$$

It suffices to prove the case $i = 0$. Given $z \in \mathrm{Trans}(X'_0, T^{r-s})$, define

$$P_0(z) \;:=\; \bigcap_{N\in\mathbb{N}} \;\bigcup_{n>N} \;\big\{z' \in X'_0 : d\big(z', T^{(r-s)n} z\big) < \eta_n\big\}.$$

It is straightforward to verify that $P_0(z)$ is a dense G_δ subset of X_0'. Moreover, given $z' \in P_0(z)$, for arbitrary $N \in \mathbb{N}$, there exists $n > N$ satisfying $d\big(z', T^{(r-s)n}z\big) < \eta_n$, which together with (4.3) yields

$$d\big(T^{sn}z', T^{rn}z\big) < \frac{1}{n}.$$

Hence,

$$\{z\} \times P_0(z) \ \subset\ \mathrm{Prox}(r,s).$$

Since $z \in \mathrm{Trans}(X_0', T^{r-s})$ was chosen arbitrarily, we have

$$\bigcup_{z \in \mathrm{Trans}(X_0', T^{r-s})} \{z\} \times P_0(z) \ \subset\ \mathrm{Prox}(r,s).$$

Due to (5) of Proposition 4.1, the density of $\mathrm{Trans}(X_0', T^{r-s})$ in X_0' and the density of $P_0(z)$ in X_0', it follows that $\mathrm{Prox}(r,s) \cap (X_0' \times X_0')$ is a dense subset of $X_0' \times X_0'$. Since $\mathrm{Prox}(r,s) \cap (X_0' \times X_0')$ is a G_δ subset of $X_0' \times X_0'$, we deduce that $\mathrm{Prox}(r,s) \cap \big(X_0' \times X_0'\big)$ is a dense G_δ subset of $X_0' \times X_0'$ and then end the proof. ☐

Proof of Theorems 1.2 and 1.3. Fix distinct $r, s \in \mathbb{N}$. Let X_i, $i = 0, 1, \ldots, k-1$ and X_i', $i = 0, 1, \ldots, k'-1$ be defined as in Lemma 4.2 and Lemma 4.3, respectively. Since $\mathrm{int}(X_0') \neq \emptyset$ and $\overline{\bigcup_{i=0}^{k-1} \mathrm{int} X_i} = X$, there is $0 \le i_0 \le k-1$ such that

$$\mathrm{int}(X_0') \cap \mathrm{int}(X_{i_0}) \neq \emptyset.$$

Denote

$$Y = \overline{\mathrm{int}(X_0') \cap \mathrm{int}(X_{i_0})}. \tag{4.4}$$

Then Lemma 4.2 and Lemma 4.3 imply that the set $W_{r,s}(\eta) \cap \mathrm{Prox}(r,s) \cap (Y \times Y)$ is a dense G_δ subset of $Y \times Y$. Since Y is perfect, it follows from Theorem 2.1 that there exists a Mycielski (r,s)-scrambled subset of Y. This completes the proof of Theorem 1.2.

We now turn to the proof of Theorem 1.3. For distinct $r, s \in \mathbb{N}$, since X is totally transitive, the values of k and k' in Lemma 4.2 and Lemma 4.3 must be restricted to 1, and hence Y in (4.4) is X. Therefore,

$$X_{r,s} \ :=\ W_{r,s}(\eta) \cap \mathrm{Prox}(r,s) \cap (X \times X)$$

is a dense G_δ subset of $X \times X$. It follows that $\bigcap_{r \neq s \in \mathbb{N}} X_{r,s}$ is also a dense G_δ subset of $X \times X$. Since X is transitive and non-trivial, it is perfect, and hence by Theorem 2.1, there exists a Mycielski set $C \subset X$ such that

$$C \times C \ \subset\ \bigcap_{r \neq s \in \mathbb{N}} X_{r,s} \ \cup\ \Delta^{(2)}.$$

Consequently, C is an (r,s)-scrambled subset of X for every distinct $r, s \in \mathbb{N}$. This completes the proof of Theorem 1.3. ☐

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